## A Three Dimensional Coordinate System for Complex Numbers

by greg ehmka

'Geometric Torus Surface Functions'

## Cover Illustration:

Shown on the cover, in cutaway, are a 'triangular' torus in a 'triangular' orbit and a 'square' torus in a 'square' orbit.

Labeling them as a ' 3,3 torus' and ' 4 , 4 torus' respectively, we can graph nearly endless possibilities. E.g. 3, 4; 6, 3; 2, 8; 8, 1 (circular); etc.

The functions are of parametric form:

$$
\begin{aligned}
& x=f(u+i v) \\
& y= \\
& i z= \\
& h(u+i v) \\
& h(u+i v)
\end{aligned}
$$

See section 14.0 5Dii: Circular Surface Functions.
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Appreciation, gratitude and celebration are expressed to literally everyone who sincerely offered articles, questions, thoughts, speculations, and even confusion relative to math on the internet. All of it is invaluable to the solitary researcher.

And, of course, many thanks to the Muses!

## Dedication:

This e-book is dedicated to all high school students, past and present, who found a spark of interest in mathematics and then decided that she or he was too math-phobic, or not smart enough, to go further. It took me nearly thirty years to write this. Don't give up on what you love!

Sincerely,
greg ehmka

## Software Note:

All of the graphs and videos in this e-book were produced with Pacific Tech's 'Graphing Calculator,' which for the most part is an inexpensive and excellent piece of software that I can completely recommend. The simple user interface is its strong point.

The difficulties arise in three places: (1) The older version videos produce AVI files which seem to come out upside down - this has been fixed in later versions; 2) The graphs are sometimes incomplete at cusps, certain vertices, and so on; and (3) At certain discontinuities, the software produces unnecessary "straight lines" as it attempts to "connect the dots" - these will be pointed out where needed.

The video files for the animations are hosted on YouTube and will play automatically when the links are clicked.

Please feel free to send along anything that might warrant attention.

## Contact Note:

In an endeavor of this size and scope, there will likely be no shortage of errors. Additionally, any innovative notations offered can best be thought of as suggestions that seemed to make sense at the time but likely can be improved upon. Corrections and thoughtful suggestions for improvement are invited - please feel free to email me.

## Navigation Note:

The List of Animations, the Table of Contents, and cross-references are 'active' allowing for easy navigation. Just click on any of them to go anywhere. If you wish to return after jumping anywhere, just click ALT + LEFT ARROW. It functions like a 'back button' on a browser.

The index is not active, so the best approach there is to type the page number into the current page box (at the top of Adobe Acrobat) and hit ENTER to go to that page. To go back type ALT + LEFT ARROW.

## Note to Mac users: You may need to manually add a 'back button' to your tool bar. To do this go to the VIEW drop-down menu. Then, click CUSTOMIZE TOOL BAR. Then, drag the FORWARD and BACK ARROWS from the menu to the tool bar for one-click forward and back navigation within this e-book.

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### 1.0 Imaginary Numbers and Human Experience

Generally speaking imaginary numbers are often thought to be, at worst, an annoyance, at times, a reluctant necessity and, at best, strange but of undeniable usefulness. At the same time there is a striving to know just what exactly an imaginary number is. A short survey of various on-line forums shows interesting discussions wherein one writer typically asks about an intuitive understanding of the imaginary unit, $i$, and other writers attempt an answer. For example, here, or here. Or if the forum is somewhat more science oriented, the discussion centers around visual/physical representations of imaginary numbers. For example, here. The source of these types of discussions stems from the wide agreement as to the algebraic definition of the imaginary unit $i$ and the perceived, by some, insufficiency as to the geometric definition of $i$.

The algebraic understanding is, of course, this definition:

$$
i=\sqrt{-1}
$$

along with comments such as this one from Leibniz and similar others:
"From the irrationals are born the impossible or imaginary quantities whose nature is very strange but whose usefulness is not to be despised. ${ }^{\left[{ }^{*}\right]}$

The current geometric understanding of $i$ is as units along the vertical axis of the complex plane along with the intuitive sense of a rotation. These are described by the Argand Diagram ${ }^{[*]}{ }^{[* *]}$ and extended via complex numbers to a circle by Euler's formula:

$$
e^{i \theta}=\cos \theta+i \sin \theta
$$

about which there is a similar duality as to wide agreement on the algebraic meaning and an insufficient understanding as to the geometric meaning. This duality is typified by comments such as this one by Benjamin Pierce:
"Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved it, and therefore we know it must be the truth."
and more recently this one by Scott E. Brodie ${ }^{[*]}$ :
> "An intuitive understanding of Euler's formula for the complex exponential, remains elusive, notwithstanding hundreds of years of contemplation."

3Di coordinates offers a significant, and as will be seen, a very satisfying advance in the geometric understanding of imaginary and complex numbers. One that implies a direct connection to human experience. As suggested in the introduction to this article, imaginary numbers can now be seen to have these two additional definitions:

In three dimensions, imaginary numbers are measurements in the 'depth' direction. Real numbers are measurements on the horizontal and vertical directions. Mathematically, depth is imaginary.

In three dimensions think of an i rotation as going from the horizontal or vertical to the front rather than from the horizontal to the vertical as in the two dimensional complex plane.

In order to establish this direct connection to human experience it is useful to note that imaginary and complex numbers have historically not been seen to have a direct connection to human experience. The philosophical concerns relative to imaginary numbers no longer generate much discussion but the direct connection to human experience is still insufficient.

This is seen in Leibniz's above quote and in these two comments by Nobel Laureate Eugene Wigner:

The complex numbers provide a particularly striking example for the foregoing. Certainly, nothing in our experience suggests the introduction of these quantities.

Surely to the unpreoccupied mind, complex numbers are far from natural or simple and they cannot be suggested by physical observations. ${ }^{[*]}$

3Di coordinates seeks to establish this direct connection to human experience beginning with some simple visualizations:

As we stand and peer out at life we "know" that we "see" a three dimensional reality. But, if we are rigorous about what we are actually seeing then we don't actually see anything except a two dimensional visual screen with an ever so slight sense of depth due to binocular vision. People who have only one eye functioning are acutely aware of the predominance of this two dimensional screen.

For example, if you walk down the street and stand squarely in front of a building you don't actually see anything other than the front of the building. What we refer to as the facade is a two-dimensional view which is even more pronounced if you close one eye.

If we don't move, and therefore, continue looking only at the front of the building through one eye, we can draw a two-dimensional coordinate system in our mind's eye upon which the front of the building could be sketched. Designers and architects do this all the time. They are sketching what one would actually see.

Mathematically, the exact point at which the eye would look, without moving, would be the origin. And, we could draw a horizontal $x$-axis and a vertical $y$-axis from that origin point.

Now, if you happened to have had your eyes closed while someone else guided you to this exact position, standing squarely in front of the building, then how would you "know" that you were looking at a whole building, rather than just the backdrop of a theater set or just the facade of a Hollywood set?

The answer, depending on how well the film or theater set was constructed, is that you would NOT know. The "rest" of the building, or whatever is behind the set, would have to be imagined.

In another example, a friend once told me of an acquaintance of his who only had one eye and who liked to play tennis. As the story goes the one-eyed tennis player had to train himself to observe the growing size of the tennis ball as it moved toward him to know the proper distance at which he should hit the ball for the return.

An even more striking example of this can be found in flight training for pilots. If a pilot is flying under VFR (visual flight rules as opposed to IFR, instrument flight rules), as another aircraft is observed, it is critical to determine whether or not there is a relative motion between the two aircraft. Here is one quote of the principle that can be found in many places:
if another aircraft appears to have no relative motion, but is increasing in size, it is likely to be on a collision course with you ${ }^{[*]}$

Quite likely some interesting thought experiments may be conducted as to under what conditions it is possible to know whether or not a tennis ball is of constant size and approaching or at a constant distance and growing in size!

We can also ask ourselves; "What is the nature of the distance that appears to be unobservable to the pilot between the two aircraft that are on a collision course?" Extending this inquiry just
a bit further; "How does the introduction of radar or a laser beam, which can and does determine the distance precisely, differ in terms of measurement from the type of measurement possible in the horizontal and vertical directions which, traditionally, uses "measuring sticks"? At the very least, a radar or laser beam requires time and a measuring stick does not.

Transferring this general inquiry directly to mathematics we might ask; "Does the real $x$, $y$ plane assume a one-eyed view? Can the common human experience of binocular vision, or in certain cases the absence of it, be suggestive of imaginary distance?

If it's true that we must imagine what goes on behind a building's facade in the $z$ or depth direction of perspective or imagine the distance to an object, moving or not, that is directly in front of us then possibly imaginary numbers can be combined with the real plane to form a new three-dimensional coordinate system more closely representative of what we actually observe.

In other words; To the degree that mathematics assists us in understanding what we perceive as we look out at life, we must acknowledge the fact that what we perceive includes both seen and imagined components. Consequently both real and imaginary values must be included in the mathematical representation of it.

As a final example, special relativity theory with its concept of length contraction in the direction of travel or parallel to it ${ }^{[*]}$ also is suggestive of a different quality to the third or depth dimension.

Although quite interesting in and of itself, how much of the foregoing discussion is factual or truthful is not what we wish to determine here. The introduction of 3Di coordinates only takes the very small step of declaring that the third or depth dimension is sufficiently different from the horizontal and vertical directions to warrant a slightly different mathematical treatment. And that slightly different treatment is only to define the third dimension as being imaginary analogous to the $y$-axis being defined imaginary in the complex plane.

As it turns out this small step of defining the third dimension as imaginary brings wonderful new results.

### 1.1 General Characteristics

The Three Dimensional Coordinate System for Complex Numbers is called '3Di' which is an acronym for 'Third Dimension Imaginary' and is based on the following definitions and characteristics:

1) In three dimensions, imaginary numbers are measurements in the 'depth' direction. Real numbers are measurements in the horizontal and vertical directions.
Mathematically, depth is imaginary.
2) In three dimensions think of an $i$ rotation as going from the horizontal or vertical to the front, rather than from the horizontal to the vertical.
3) A 'dimensional approach' is taken to functions. Meaning, functions are categorized by the number of dimensions the particular function has. By 'dimension' what is meant is essentially a variable.
4) One of the primary points of view is that imaginary dimensions are as equally important, and as equally present, as real dimensions.
5) The primary purpose of the e-book is to present a wonderful array of the many new functions made possible by '3Di' and to graph these functions.

To make it a little clearer, with this model we could say that the usual real plane might be designated as '2D,' the complex plane as '2Di,' the usual real space as 3D, and this system as 3Di.

### 1.2 Associated Dimensions and Functions

Functions may be categorized by the number of dimensions i.e. variables. For this categorization, the presence and number of $i$ 's in the designation denotes the number of imaginary dimensions or imaginary variables. Continuing from 2D, 2Di, 3D, 3Di we may have for example, 4Di, 4Dii, 5Dii, 6Diii and so on.

The functions that are graphed, therefore, associate the number of variables - including both input and output - to the number of dimensions and give rise to the following useful categorization of types of functions:

3Di $\quad y+i z=f(x)$

4Di

$$
y+i z=f(x, t)
$$

4Dii $\quad y+i z=f(x, i t)$

4Dii $\quad y+i z=f(x+i \theta)$

4Dii $\quad y+i x=f(\theta+i z)$

4Dii $\quad y+i z=f(u+i v)$

5Dii $\begin{aligned} & x=f(u+i v) \\ & y=g(u+i v) \\ & i z=h(u+i v)\end{aligned}$
$6 \mathrm{Dii} \quad \begin{aligned} & x=f(u+i v, t) \\ & y=g(u+i v, t) \\ & i z=h(u+i v, t)\end{aligned}$
helixes, polynomials, conics, elliptic and hyperelliptic curves
function morphing
helix morphing
complex natural logs
complex imaginary logs
closed and open surfaces
circular surfaces
closed surface objects in motion

With five and six variables, $x$ becomes exclusively an output variable, whereas, in lower dimensions it is usually both an input and an output dimension.

### 1.3 Example in 4Dii

As can be seen in the Table of Contents, this eBook organizes all of the many and varied functions by their number of variables consistent with the table above. Here is an example of 4Dii which generates a new interpretation of the exponential/natural log functions:

$$
\begin{gathered}
y+i z=e^{x+i \theta} \\
\ln (y+i z)=x+i \theta
\end{gathered}
$$

In 4Dii these form a simple geometry of a rotation of the usual exponential graph. This way it can be seen that the 'infinite branches' of the log function, which show up in two or three dimensions, can be interpreted geometrically as rotations of the three-dimensional exponential graph, with $i \theta$ specifying the amount of rotation. In this way, the multi-valued nature of the natural log function is simply the state of rotation of the three-dimensional exponential graph.

3Di coordinates allow the complex exponential function and its inverse, the complex natural logarithmic function, to be displayed as a four-dimensional function with a wonderful geometric interpretation.


## Animation 1 'Rotating Exponential Graph'

Any point on the graph, at any state of rotation, is uniquely specified by the four coordinates ( $x, y, i z, i \theta$ ) as follows:
$x$ is the real horizontal axis coordinate.
$y$ is the real vertical axis coordinate.
$i z$ is the imaginary depth axis coordinate.
$i \theta$ is the imaginary rotation of the exponential graph.

This interpretation also shows why the natural log of zero, or of $(y, i z)=(0,0 i)$, is not possible since any point with these coordinates is not on the graph. Either $y$ or iz may be zero, but not both.

Additionally, changing the input from 'line-angle input', $(x+i \theta)$, to 'complex regional input', $(u+i v)$ :

$$
\begin{gathered}
y+i z=e^{u+i v} \\
\ln (y+i z)=u+i v
\end{gathered}
$$

forms a surface by, graphing a complete revolution at once where any point on the surface is specified by the four coordinates:


See section 11.23 The Four Coordinate Complex Exp/Log Surface

As the eBook progresses, moving from one category of functions to the next, when the number of dimensions/variables becomes greater than four (which may be thought of as the four observable dimensions), we can formulate a notion of embedded dimensions that play more of a parametric role.

For example, in 6Dii:

6Dii

$$
\begin{aligned}
& x=f(u+i v, t) \\
& y=g(u+i v, t) \\
& i z=h(u+i v, t)
\end{aligned}
$$

there would be the usual three spatial dimensions, $x, y, i z$ plus the usual motion or animation dimension indicated by 't.' These four would be the observable dimensions. And then two more dimension indicated by, in this case, $u+i v$ which would be the embedded dimensions playing a parametric role.

The point in this modeling, again, is that dimensions would be characterized by the number of variables, rather than by any intrinsic properties of a space or an object. See 13.1 Observable and Embedded Dimensions for more discussion.

### 2.0 3Di Means: "Third Dimension Imaginary"

### 2.1 Constructing the Coordinate System

As we look at the building's façade with our eye on the origin, we can make what we see correspond to the real plane. Accordingly, we could then specify functions on this real plane of the type:

$$
y=f(x)
$$

The horizontal axis takes on the values of $x$, and the vertical axis takes on the values of $y$. And, in accordance with engineering design (as well as everyday usage), we will call this the 'front view.' Further, since it is a real plane we will call it the 'front real plane' or FRP. In this modeling we can designate it as 2D.

Behind the building's façade is, of course, the third-dimension, and we give that dimension the variable $z$ - as the depth axis. We can designate this as 3D.

So, we have thus far:

$$
\begin{gathered}
x=\text { the horizontal axis } \\
y=\text { the vertical axis } \\
z=\text { the depth axis }
\end{gathered}
$$

This is different than that which is used in some other applications, particularly surfaces, where $z$ is often used for the vertical direction.

Mathematically, the problem is how to specify functions going on "behind the façade" since we cannot see them. Conveniently, mathematics has another tool called the "complex plane" which uses imaginary numbers.

This plane uses the following:

$$
\begin{gathered}
x=\text { the horizontal axis } \\
\text { iy }=\text { the vertical axis }
\end{gathered}
$$

This plane is usually called $C$, but we will designate it as 2Di. Here is a clue:

If it's true that we must imagine what goes on behind the façade in the $z$ or depth direction of perspective, then possibly imaginary numbers can be combined with the real plane to form a new three-dimensional coordinate system.

So then, the problem would be how to combine the real plane with the complex plane?

In the complex plane the real values are in the horizontal direction. This is the same as the real plane. So far so good. And, in the complex plane, the imaginary numbers are in the vertical direction, whereas we want them in the depth direction. So, if we take the complex plane, as we look at it, and then lay it down flat in front of us - effectively rotating it, top-end forward, about the horizontal axis - then the imaginary numbers will be in the forward-backward, or depth direction , rather than the vertical. Then, by changing the axis name for the complex plane from $i y$ to $i z$, for the depth direction in our new three-dimensional system, we have the new coordinates and we can designate this as 3Di.

This would be the same as taking our initial $\mathrm{x}, \mathrm{y}, \mathrm{z}$ space and multiplying the z by i togive iz in the depth direction.

This combined real and imaginary plane construction allows us to write functions of the type:

$$
y+i z=f(x)
$$

in which:
i. $\quad$ The real input $x$ will be graphed on the horizontal axis.
ii. The real output $y$ will be graphed on the vertical axis.
iii. And, the imaginary output $i z$ will be graphed on the "depth" axis.

So, in step-by-step progression from 2D to 2Di to 3D to 3Di, we have made the transition from two dimensions to three and combined the real and complex planes - with our eye now at the origin of three axes instead of two.

But wait a minute. Doesn't the equation we just wrote normally mean only curves on the real plane since: $z=0$ ?

Well, the answer is: it depends. If $f(x)$ only generates real numbers like these functions:

$$
f(x)=x^{2} \text { or } f(x)=e^{x}
$$

Then yes, $z$ would always equal zero, and so $i z$ would equal zero, and - very important - our function would only exist on the front real plane (FRP.) But take, for example, a function like:

$$
y+i z=i x
$$

Then $y$ would always equal zero rather than $z$. And, we can ask the question: Where would this straight line lie?

Since $y$ (the vertical axis) is always zero, this line lies entirely on the ( $x, i z$ ) plane, which is the horizontal and depth plane. This plane cannot be seen by our eye at the origin of the front view since it extends directly forward and backward of our eye. We will call this plane the 'top view.' And since it is an imaginary plane we will call it the 'top imaginary plane' or the TIP.

In order to see this plane we would have to look down on it. And if we did, then the straight line would look like:


[^0]And, in three dimensions:


3D straight line with vertical component zero

So the function:

$$
y+i z=f(x)
$$

can produce three-dimensional space functions that have imaginary part zero, and lie only on the front real plane, or functions that have real part zero, and lie only on the top imaginary plane. By extension, $f(x)$ can produce complex numbers that lie neither on the front real, nor the top imaginary planes, but anywhere in space. As we shall see, they may also lie partly on the front or top planes as well.

Now consider the three-dimensional space function formed by the Euler Formula:

$$
y+i z=e^{i x}=\cos x+i \sin x
$$

in which:

$$
\begin{gathered}
x=\text { the horizontal axis } \\
y=\text { the vertical axis }=\cos x \\
\text { iz }=\text { the depth axis }=i \sin x
\end{gathered}
$$

This gives the result that, in three dimensions, Euler's Formula passes the vertical line test for functions. Euler's formula becomes a helix extending left and right along the horizontal axis. Its real value output gives the amplitude in the vertical direction and it's imaginary values give the amplitude in the depth direction.


The "Euler Helix"

More on Euler's Formula and Helix in section 6.0 Euler's Formula Upgraded, Helix and Spiral Functions

And as we will see throughout this book, one of the advantages of 3Di is that $f(x)$ itself may be treated as a three-dimensional function and generate complex number output from real, imaginary or complex input, and by so doing, allow all three dimensions to be treated in one equation.

### 2.2 A Left Hand System

The positive directions are right, up, and forward (the direction we are looking). And, this makes a left hand coordinate system in which, from the origin:
I. thumb pointing up - the vertical axis - takes on positive real ' $y$ ' values
II. index finger pointing forward - the depth axis - takes on positive imaginary ' $z$ ' values
III. middle finger pointing right - the horizontal axis - takes on positive real ' $x$ ' value

3Di is a left hand coordinate system:


### 2.3 Projection Planes

With any given 3Di space function, $y+z i=f(x)$, there will be six resulting two-dimensional 'projection planes' which are the two sides of the three colored planes shown below.


Of the six resulting two-dimensional planes, two of the planes are 'real planes' and four of the planes are 'imaginary planes', as follows:
i. Both sides of the violet plane - horizontal axis (real) and vertical axis (real) $(x, y)$, which we will call the front and rear real planes.
ii. Both sides of the red plane - horizontal axis (real) and depth axis (imaginary) ( $x, i z$ ), which we will call the top and bottom imaginary planes.
iii. Both sides of the blue plane - depth axis (imaginary) and vertical axis (real) (iz,y), which we will call the left and right side imaginary planes.

In practice we will rarely, if ever, use the rear and bottom planes and only occasionally use the left side plane.

Front real plane $(x, y)$, projected from $y=\operatorname{real} f(x)$ :


Top imaginary plane, $(x, i z)$, projected from $\operatorname{iz}=\operatorname{imag} f(x)$ :


Right side imaginary plane, $(i z, y)$, projected from $\begin{gathered}i z \\ y\end{gathered}=\begin{gathered}\operatorname{imag} f(x) \\ \operatorname{real} f(x)\end{gathered}$ :


Going back to the idea of standing in front of a building with one eye closed, what we actually "see" is the 'front real plane' or $F R P$. The top view and the right side view must be imagined, and therefore we call them the 'top imaginary plane', or TIP , and the 'right side imaginary plane', or RSIP.

What we are asserting is that to the degree that mathematics, and geometry in particular, assists us in understanding what we perceive as we look out at life, we must acknowledge the fact that what we perceive includes both seen and imagined components. Consequently both real and imaginary values must be included in the mathematical representation of it.

### 2.4 Real, Imaginary or Complex Function Input and Output

The terms 'real function', 'complex function', 'complex valued function', 'complex valued function of a complex (or real) variable', and so on, have different definitions by different writers.

One difference being that for a 'complex function' the range only is defined to be complex by some , while the range and domain are defined to be complex by others. This is further complicated when one attempts to graph the various possibilities and needs to allow for 'four dimensions.'

Using a simple logic table we have these possibilities:

$$
\text { output }=f(\text { input })
$$

| Input $\rightarrow$ <br> Output <br> $\downarrow$ | Real Only | Imaginary Only | Complex |
| :--- | :---: | :---: | :---: |
| Real Only | $y=f(x)$ | $y=f(i w)$ | $y=f(x+i w)$ |
| Imaginary Only | $i z=f(x)$ | $i z=f(i w)$ | $i z=f(x+i w)$ |
| Complex | $y+i z=f(x)$ | $y+i z=f(i w)$ | $y+i z=f(x+i w)$ |

In 3Di, whether the output is complex, imaginary, or real isn't so important in and of itself because what this means on the graph is that the output point is either in space, on one of the imaginary planes, or on the real plane respectively.

Further, when we consider the possibilities of a complex input and complex output we do, in fact, have four dimensions. Additionally, this fourth-dimension can be real or imaginary as in the following two examples:

$$
\begin{gathered}
y+i z=f(x, t) \\
y+i z=f(x+i w)
\end{gathered}
$$

In the first we have three real dimensions and one imaginary dimension. In the second we have two real dimensions and two imaginary dimensions. These may be delineated as 4Di and 4Dii respectively.

What this means is that a function with real input and real output will show up only on the front real plane (FRP.) Real input and imaginary output will show up only in the imaginary top view on the TIP. Real input and complex output will show up in space and so on. If a three- or four- dimensional curve is projected to one of the two-dimensional planes, only two of the three (or four) values will be graphed, and this will depend on which plane the curve is being projected to. In the case of four dimensions there is the additional choice of which input values to graph.

With five dimensions:

$$
y+i z=f(x+i w, t)
$$

There are that many more choices of which "view" to graph.

### 2.5 Projecting a Space Curve

To further illustrate how a three-dimensional space curve is projected to the front, top and right side planes: place your left thumb, index finger, and middle finger in initial position, and then:
i. For the two dimensional projection view of the 'front real plane' (FRP), hold your fingers and thumb in initial position pointing the index finger forward, thumb up and middle finger right. Then, raise the hand to eye level until you cannot see the index finger. The intersection of the thumb and middle finger is the origin of the FRP.
ii. For the two-dimensional projection view of the 'top imaginary plane' (TIP), which has real values for the horizontal axis and imaginary values for the vertical axis, hold your fingers in initial position and rotate towards you (around the middle finger axis) such that you point the index finger up and the thumb towards you. The intersection of the index and middle fingers is the origin of the TIP.
iii. For the two-dimensional projection view of the 'right side imaginary plane' (RSIP) , hold your fingers and thumb in initial position and rotate clockwise (CW) around the thumb such that you point the middle finger at yourself. The intersection of the thumb and index finger is the origin of the RSIP.

Now to bring this all together: If we take an arbitrary space curve, which, as we will see later, has the 3Di equation $y+i z=e^{12 i x}+x$ :

Here is the 3Di graph:



Animation 2 'Space Curve'

Projecting to the FRP, $y=\operatorname{real}\left(e^{12 i x}+x\right)$ :


Projecting to the TIP, $i z=\operatorname{imag}\left(e^{12 i x}+x\right)$ :


And projecting to the RSIP, $\quad \begin{aligned} & i z \\ & y\end{aligned}=\begin{aligned} & \operatorname{imag}\left(e^{12 i x}+x\right) \\ & \operatorname{real}\left(e^{12 i x}+x\right)\end{aligned}$ :


And then, combining all three projection views, FRP in violet, TIP in red, RSIP in blue:


See section 6.0 Euler's Formula Upgraded, Helix and Spiral Functions for more on the Euler Helix and other helixes and spirals.

### 2.6 Conics in 3Di

In the coordinate system '3Di', the normal conic hyperbola and normal conic ellipse are two 2D views (orthogonal to one another) of the same 3D object!

To illustrate the basic idea here is the usual equation of a hyperbola:

$$
\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=C
$$

And, with $\mathrm{a}=\mathrm{b}=1$ and $\mathrm{C}=1$,

$$
y= \pm \sqrt{x^{2}-1}
$$

And, the normal graph in 2D:


If we take another look at the graph, and this time ALSO incude the interval $-1<x<1$ :


The graph line joining the vertices actually graphs the real value of $y$, which is zero at each point, if the domain of $x$ is allowed to take on values between 1 and -1 .

With $C=1$, in a normal 2D graph, there is a gap between the two vertices in the $x, y$ plane (FRP, x is horizontal, y is vertical). This is because the values for x between the two vertices, if inserted into the equation, produce complex numbers and are therefore not usually shown. Now that we have an interpretation for these numbers, and allow them as part of the domain, what shows up in three dimensions is a circle in the $x, i z$ plane (TIP, $x$ as horizontal, and $i z$ as vertical) in between the vertices!

The associated ellipse/circle in TIP:


And then, seeing the three dimensional view:

"Conic" Hyperbola With Orthogonal Circle


Animation 3 '3Di Conic'

The TIP circle is orthogonal to the FRP hyperbola, and is not visible in the Front Real Plane (FRP) view, because the normal 2D plane graphs are projections. The circle only becomes visible in a 3D view, or when viewing the Top Imaginary Plane (TIP) directly, that is, in top view.

See: Section 4.1 Quadratic Input and Output for further discussion of the vertices and 'bifurcation.'

Additionally, notice that when square roots are taken, two three-dimensional functions are generated: one for each root. See next section.

### 2.61 Conic Nonlinearity

The term 'nonlinearity' here means that there is an exponent other than one on the dependant variable. In the above hyperbola-ellipse example, when we take the two square roots, if complex numbers are generated, there are actually four values output for each single input. These are the real and imaginary parts of each root.

So, the hyperbola-ellipse equation:

$$
\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=C
$$

with $\mathrm{a}=\mathrm{b}=1$ and $\mathrm{C}=1$, and rearranged, is:

$$
y^{2}=x^{2}-C
$$

and, in 3Di becomes:

$$
\begin{gathered}
(y+i z)^{2}=x^{2}-C \\
y+i z= \pm \sqrt{x^{2}-1}
\end{gathered}
$$

In this form there are two output values - one real and one imaginary - for each root (function) produced for each input value.

Both of these roots are then graphed in accordance with the 3Di axes: The input $x$ being on the horizontal axis. (Left middle finger, pointed right.) The real output $y$ being on the vertical axis. (Left thumb, pointed up.) And, the imaginary output $i z$ on the 'depth' axis. (Left index finger, pointed forward.)

This then, gives the two separate functions generating two three-dimensional space curves:
$y+i z=+\sqrt{x^{2}-1}$ in red:
$y+i z=-\sqrt{x^{2}-1}$ in blue:


As we go on, we shall see that whenever roots are taken, multiple 3Di functions are generated. This provides a tool for the analysis of non-linearity; meaning exponents on the dependant variable of any given function.

The two functions given by:

$$
y+i z= \pm \sqrt{x^{2}-1}
$$

by adding the notation of 'Demoivre Numbers',

$$
a+b i=e^{2 \pi i \frac{k}{n}}
$$

where $k$ is the $k^{t h}$ root of the $n$ roots of unity, these two functions can be written,

$$
y+i z=e^{2 \pi i \frac{k}{n}}\left(x^{2}-C\right)^{\frac{1}{2}}
$$

in which $e^{2 \pi i \frac{k}{n}}$ means both second roots of 1 . And individually:

$$
\begin{aligned}
& e^{2 \pi i \frac{1}{2}}=-1, \text { the first root } \\
& e^{2 \pi i \frac{2}{2}}=1, \text { the second root }
\end{aligned}
$$

So, with a third degree exponent on $y$ :

$$
y^{3}=x^{2}-C
$$

becomes:

$$
(y+i z)^{3}=x^{2}-C
$$

And, $e^{2 \pi i \frac{k}{n}}$ means all three cube roots of 1 .

$$
y+z i=e^{2 \pi i \frac{k}{n}}\left(x^{2}-C\right)^{\frac{1}{3}} \text { with } n=3 \text { and } k=1,2,3
$$

And, individually:

$$
\begin{gathered}
e^{2 \pi i \frac{1}{3}}=\frac{-1+i \sqrt{3}}{2}, \text { the first cube root of } 1 \\
e^{2 \pi i}=\frac{-1-i \sqrt{3}}{2}, \text { the second cube root of } 1 \\
e^{2 \pi i \frac{3}{3}}=1, \text { the third cube root of } 1
\end{gathered}
$$

with: $C=4$ and interval $-6 \leq x \leq 6$ (first root in blue, second in red and third in black), note that the segments of the three functions that are on the FRP are the upper half hyperbola part of the black graph, and the lower ellipse part of the blue graph.


[^1]
## The third cube root graph (black):



The first cube root graph (blue):


The second cube root graph (red):


For any degree $n$ on the dependant variable:

$$
y^{n}=x^{2}-C
$$

the individual roots functions are:

$$
y+i z=e^{2 \pi i \frac{k}{n}}\left(x^{2}-C\right)^{\frac{1}{n}}
$$

in which $e^{2 \pi i \frac{k}{n}}$ are all of the $n t h$ roots of 1 for any degree $n$. In section
4.59 Lame Curve Bifurcation, we will see that the exponent on $x$ may be any degree also.

See also: section 7.1 Helix and Spiral Nonlinearity, section 4.11 Quadratic Nonlinearity, and section 4.65 Polynomial Nonlinearity.

### 2.7 Helixes in 3Di

The simplest helix graph is generated by Euler's equation in 3Di coordinates:

$$
y+i z=e^{i x}
$$

Interval $-4 \pi \leq x \leq 4 \pi$ :


A simple coefficient $f$ in the exponent determines the frequency:

$$
y+i z=e^{f i x}
$$

with $f=12$ :


As we go through the sections of the book, it becomes clear why Euler's Formula is regarded as the most remarkable equation in mathematics. It is innate almost everywhere.

See section 6.0 Euler's Formula Upgraded, Helix and Spiral Functions for more discussion.

### 2.71 Beat Helix

By adding a second term with a frequency close to the first one, a 'Beat Helix' results:

$$
y+i z=e^{f x i}-e^{g x i}
$$

with $f=20, g=18$ :

FRP


TIP


RSIP, interval $-6<x<6$ :


Three plane projection with $f=20, g=18$, interval $-6<x<6$ :


3Di object with $f=20, g=18$, interval $-6<x<6$ :


Animation 5 'Beat Helix'

As we shall see in section 3.0 Historical Curves in 3Di, simple sums of helixes with different frequencies and amplitudes will produce, in side view, literally dozens of the classical curves of history. Here is an example that resembles one in the Epicycloid/Hypocycloid family:


### 2.72 Combining Bases

Virtually any algebraically valid statement can be effectively graphed with no special significance given to multiple bases, be they the same or different.

As a simple example, consider the equation:

$$
y+i z=(\cos \mathrm{x}) *\left(i^{x}\right)
$$

which is equivalent to:

$$
y+i z=\left(\text { real } e^{x i}\right) *\left(i^{x}\right)
$$

More on the $i$-base helix in section 6.51 The $i$-base Exponential. In a simple way the two bases interact such that the two separate periodicities produce one helix:
with interval $-4 \pi \leq x \leq 4 \pi$, here is the FRP (the real part $y$ in violet) and the TIP (the imaginary part $i z$ in red) superimposed:


And then, superimposing the RSIP $(i z, y)$ with $x$ ungraphed in blue and zooming in:


In 3Di with $-12<x<12$ :


Animation 6 'Two Base Helix

### 2.8 Exponentials in 3Di

### 2.81 Exponential Graph Rotation

As seen in section 1.3 in four dimensions beginning here with two dimensions, the usual exponential function that shows up in two dimensions has equation:


In 3Di this equation would be:

$$
y+i z=e^{x}
$$

with $z=0$ for all values of $x$. If we were to add an arbitrary imaginary constant, say $1.5 i$, to the exponent:

$$
y+i z=e^{x+1.5 i}
$$

the two dimensional graph, i.e. the FRP projection in red, would then be:


The two dimensional RSIP (right side imaginary plane) projection for this red graph is:


And in three dimensions:
Normal exponential graph in black.
Normal exponential graph with imaginary constant, $1.5 i$ added to exponent, in red.
greg ehmka, 2013


The effect of adding the imaginary constant to the exponent is to rotate the graph in the positive imaginary, or depth dimension.

In three dimensions think of an 'i' rotation as going from the horizontal or vertical to the front, rather than from the horizontal to the vertical.

This forms the basis of a new geometric interpretation of complex logarithms. See section 6.52 Rotating Exponentials.

### 2.82 Inverse Lambert W

Extending this notion of exponential graph rotation to, for example, the Inverse Lambert W function:

$$
f(w)=w e^{w}
$$

with:

$$
w=x+i \theta
$$

if we were to add an imaginary constant, e.g.:

$$
\theta=\frac{\pi i}{2}
$$

the equation becomes:

$$
f(w)=\left(x+\frac{\pi i}{2}\right) e^{x+\frac{\pi i}{2}}
$$

and in 3Di coordinates:

$$
y+i z=\left(x+\frac{\pi i}{2}\right) e^{x+\frac{\pi i}{2}}
$$

then the three dimensional graph, in blue, is altered to:


This blue graph in TIP (top Imaginary plane) projection with coordinates ( $x, i z$ ) has the graph:


So, this is the same graph projected to the TIP and so appears in the top view rather than on the real plane. Other values for theta will rotate the graph just as with the usual exponential function.

### 3.0 Historical Curves in 3Di

A very large number of the historical curves, especially those that are inherently circular in nature, can be produced by combining two or more helixes and then projecting the result to the side view.

### 3.1 The Elliptic Helix

The elliptic helix is the result of a sum or difference of two helixes with the same frequency that are reciprocals of one another with different amplitudes.

The difference between a helix and its reciprocal is the development: CCW (counter clockwise) for the helix and CW (clockwise) for its reciprocal.
$y+i z=e^{i x} \quad$ in gray;
$y+i z=\left(e^{i x}\right)^{-1} \quad$ in aqua;


Animation 7 'Reciprocal Helixes'

If the helix and its reciprocal are added, depending upon signs, either the imaginary parts or the real parts will cancel leaving only a plane sine wave. Consequently, to obtain the elliptic helix the amplitudes must be unequal.
And so, the equation for the elliptic helix is:

$$
y+i z=A e^{i x}+B\left(e^{i x}\right)^{-1} \quad A \neq B
$$

And, in the planar view, RSIP (Right Side Imaginary Plane,) a simple addition or subtraction of the two amplitudes gives the semi-major and semi-minor axis lengths for the projected ellipse.

$$
\begin{aligned}
& A+B=\text { semi major axis } \\
& A-B=\text { semi minor axis }
\end{aligned}
$$

In 3Di, with: $A=1$ and $B=2$, and interval $-4 \pi \leq x \leq 4 \pi$ :


And in RSIP, (hor., vert.) $=(i z, y)$ :


### 3.2 The 'Cusped Helix'

Although we would not normally think of it as such, the elliptic helix is actually a 'two cusped' helix. As we shall see this is a consequence of the frequencies being equal. As we begin to alter the frequencies of the two helixes, an amazing array of remarkable results occurs.

For example, leaving amplitudes and reciprocals the same, if we now insert additional coefficients, $a$ and $b$, such that they become: $a=2$ and $b=1$, the two frequencies add to give the number of 'cusps.'

### 3.21 Tricuspoid

$$
y+i z=A e^{a i x}+B\left(e^{b i x}\right)^{-1}
$$

The result is a 'Tricuspoid Helix' and a corresponding Tricuspoid in RSIP:



Animation 8 'Tricuspoid Helix'

### 3.22 Astroid

With $\quad y+i z=A e^{a i x}+B\left(e^{b i x}\right)^{-1} \quad a=3, b=1, \quad a+b=4$, and in this case, changing amplitudes to $B=3$ :


Animation 9 'Astroid Helix'

### 3.23 5-cuspoid, Etc.

With $y+i z=A e^{a i x}+B\left(e^{b i x}\right)^{-1}, \quad a=4, b=1, \quad a+b=5$
and, in this case, changing amplitudes to: $B=5$, a 'pentagon' results which may have more or less straight sides depending on amplitudes.

Showing the RSIP on the left and, this time, the three planar projections separately on the right:


There is a subtle difference if the frequencies are changed to $a=3, b=2$ even though the sum is, as above, still $a+b=5$. The difference is there are now five 'loops' rather than cusps.

Continuing with $B=5$, $f$ ull helix on the right:



And, adjusting amplitude to $B=1.5$ :


Needless to say the possibilities are virtually limitless.

This principle can be applied in many situations. See section 3.46 Square Cornu Spiral/Fresnel Integrals, and section 14.3 Geometric Torus Surfaces, and section 6.54 The Elliptic Spiral.

### 3.24 Trifolium

The 'cusps' may become 'loops' with adjustment of amplitudes. So returning to 'Tricuspoid frequency' settings:
with:

$$
y+i z=A e^{a i x}+B\left(e^{b i x}\right)^{-1} \quad a=2, b=1, \quad a+b=3
$$

and, in this case, changing amplitudes to: $B=1, A=1$
the result is the Trifolium:


### 3.25 Rose Curves, Quadrifolium, Etc.

The following coefficients produce the Quadrifolium in RSIP. The Quadrifolium is often shown with a $\pi / 4$ rotation relative to these figures.

For a simple rotation of any RSIP view, the coefficient $i^{d}$ can be used as a 'rotator':

$$
y+i z=i^{k}\left(A e^{a i x}+B\left(e^{b i x}\right)^{-1}\right)
$$

With $k=0$ :


With $a=3, b=1, A=B=1, k=.5$ :


Adjusting amplitude,


$$
B=6: \quad B=-6
$$




Rose Curves of any number of 'petals' may be produced by adjusting the coefficients $a$ and $b$.
There are harmonics involved. For example, $a=6, b=2$, although adding to eight produces four petals/loops/cusps instead.

Larger frequencies produce larger numbers of loops, cusps, etc.

With $a=23, b=2, \quad a+b=25$ and $B=1$ :


And, adjusting amplitude with $B=2$ :


### 3.26 Epicycloid, Hypotrochoid, Epitrochoid, Hypocycloid

The differences between these four types of curves become somewhat indistinct as the frequencies and amplitudes take on different values. The values for $a, b, A, B$ can be positive or negative integers and any intermediate value. All coefficients are valid producing a true infinity of results. The harmonics of $a$ and $b$ also apply producing additional results.
E.g., $a=\frac{3}{4}, b=\frac{2}{3}, A=B=2$ :


With different frequency signs, $a=3, b=-2, B=1.5$, the interval continues to be $4 \pi \leq x \leq 4 \pi$ :



With $a=8, b=2.5, B=2$ :


With $a=8, b=3, B=1.15$ :

greg ehmka, 2013

### 3.3 Cycloid

The usual parametric equation of a linear cycloid:

$$
x=a t-h \sin (t), y=a-h \cos (t)
$$

In 3Di helix equations, sine curves and cosine curves are just the imaginary and real parts of the helix. So, if we begin with a sum of a helix and its reciprocal, and keep the amplitudes the same, then the imaginary parts will cancel, leaving us the real part:

$$
y=e^{i x}+e^{-i x}
$$

This gives us what in the parametric set of equations would be the cosine portion, which, being real, shows up in the FRP (violet.)

Interval $-6 \pi \leq x \leq 6 \pi$ :


Similarly the difference of a helix and its reciprocal will cancel the real parts leaving us the imaginary part which is orthogonal to the real part.

$$
i z=e^{i x}-e^{-i x}
$$

This gives us what, in the parametric set of equations would be the sine portion, which being imaginary, shows up in the TIP (red):


If we then combine these into a single equation:

$$
y+i z=\left(e^{i x}+e^{-i x}\right)+\left(e^{i x}-e^{-i x}\right)
$$

and then, operate on the two separate parts in accordance with the parameters of a linear cycloid:

$$
y+i z=\left(a x-h\left(e^{i x}+e^{-i x}\right)\right)+\left(a-h\left(e^{i x}-e^{-i x}\right)\right)
$$

the resulting helix, $a=h=1$ :


If we then deconstruct the helix into its three planar projections, $a=h=1$ (FRP in violet, TIP in red, and RSIP in blue):

and then, isolate the RSIP view in two dimensions, with: $a=h=1$ :


Changes to ' $a$ ' will change the slope of the helix. With $a=1$, the value of ' $h$ ' produces the three different types of cycloid. The cycloid, curtate cycloid, and prolate cycloid.

With $h>.5$ (the prolate cycloid pictured above) and with $h<.5$ (here: $h=.3$ ), the curtate cycloid :


With $h=.5$ we have the usual cycloid:


Notice that the cycloid is in the RSIP rolling along the vertical line; $i z=i$.

If we desired to clean it up for normal presentation - meaning have the linear cycloid rolling along the $x$-axis in the usual real plane - we would:
switch coordinates with:
2D $x=3 D i y$
$2 D y=3 D i i z$

adjust; 'a' to -1:

translate in both directions:
$x=y+1+\frac{\pi}{2}$
$y=i z+1$


### 3.31 Imaginary Slope

All of the detail we just applied to arrange the helix to match the historical Cycloid is to provide a larger context for the actual 3Di equation of the Cycloid. In 3Di coordinates the Cycloid is also a RSIP projection like many of the other historical curves but the 3Di equation includes the concept of imaginary slope.

As we saw previously, here is the graph of a line with equation:

$$
y+i z=i x
$$



This line lies entirely on the TIP and is not visible in the FRP except as a line that coincides completely with the $x$-axis. The rotation that would bring this line to its present position can be thought of either as an aircraft changing its heading or as a 'yaw' of a spacecraft. This change of aircraft heading or spacecraft yaw is 'imaginary slope.' In section 5.0 Complex Slope, the geometry and algebra of complex slope is presented in detail.

If we then take this imaginary sloped line and add it to a helix:

$$
y+i z=e^{i x}+i x
$$

The following graph results. The above line is included for reference:


The RSIP projection of this helix is then:


The RSIP projection will match the historical curve with an upward translation and an $i \pi$ phase shift:

$$
y+i z=e^{i x+i \pi}+i x+1
$$



If we add a variable imaginary slope $m$ which will determine different values for the imaginary rotation of the helix:

$$
y+i z=e^{i x+i \pi}+i m x+1
$$

These different imaginary slopes will project to the RSIP the different forms of the Cycloid. E.g. the prolate cycloid, in red with $m=.5$, the curtate cycloid, in black with $m=1.5$ etc:


### 3.4 Historical Spirals in 3Di

### 3.41 Archimedes Spiral

$$
r=\theta
$$

And, in 3Di, the Archimedes spiral is just the RSIP of:

$$
y+i z=x e^{i x}
$$

but, with a reversal of the horizontal and the vertical axes. The historical curve is on the left and the RSIP curve is on the right.

Interval $0 \leq x \leq 12 \pi$ :

Historical


RSIP


Superimposing the two graphs shows that they are symmetrical about the line $y=x$;


The RSIP (right side imaginary plane) has the imaginary values from any function on the horizontal and the real values from any function on the vertical. So, the historical curve is the equivalent of reversing this; that is, graphing the real values on the horizontal and the imaginary values on the vertical.

If the interval is extended in the negative direction as well, with interval $-12 \pi \leq x \leq 12 \pi$ :


Once again, the heart shaped loops, cardioids, and variations thereof are occurring over and over.

Here is the FRP on the left and the spiraling helix on the right:



### 3.42 Fermat's Spiral

$$
r= \pm a \theta^{\frac{1}{2}} \quad a=1
$$

Fermat's spiral is the RSIP of;

$$
y+i z= \pm x^{\frac{1}{2}} e^{i x}
$$

With the same discussion as to isolating the RSIP and then reversing the graphing axes, meaning a change of coordinate systems from 2D to 3Di with:

2D $x=3 D i y$
$2 D y=3 D i i z$

Interval $0 \leq x \leq 12 \pi$ :

Historical RSIP


As previously shown, due to the graphing of the two square roots, this is actually two helix functions. Extending the interval to include negative $x$, the two helixes (positive root in aqua and negative root in yellow), plus the side view projection of all four pieces, (positive and negative roots for plus and minus ' $x$ '):

Interval $-12 \pi \leq x \leq 12 \pi$ :


Animation 10 'Fermat's Spiral Helix'

### 3.43 Hyperbolic Spiral

2D Polar equation:

$$
r=a \theta^{-1} \quad a=1
$$

3Di Helix equation:

$$
y+i z=x^{-1} e^{i x}
$$

With the same discussion as to isolating the RSIP, and then reversing the graphing axes with:
$2 D x=3 D i y$
2D $y=3 D i z$
interval $0 \leq x \leq 12 \pi$ :

## Historical



RSIP


Here is the helix with an increase in frequency ' $f$ ' and an increase in amplitude ' $A$ ' to show the graph more clearly:

$$
y+i z=A x^{-1} e^{f i x} \quad A=3, \quad f=3
$$

Interval $0 \leq x \leq 6 \pi$ :


The top view (TIP) of the hyperbolic spiral has an interesting graph. If we extend the interval to include negative ' $x$ ' the graph is discontinuous at zero, periodic in both directions, and shows a decreasing amplitude. It also shows a constant frequency with period equal to $\frac{2 \pi}{f}$.

$$
y+i z=A x^{-1} e^{f i x} \quad A=2, \quad f=2
$$

TIP with interval $-6 \pi \leq x \leq 6 \pi$ :


### 3.44 Lituus

2D Polar equation:

$$
r= \pm \theta^{\frac{-1}{2}}
$$

3Di helix equation:

$$
y+i z= \pm x^{\frac{-1}{2}} e^{i x}
$$

With the two coordinate systems related as before:
$2 D x=3 D i y$
2D $y=3 D i i z$

Graphing both roots with interval: $0 \leq x \leq 9 \pi$

2D Polar
3Di RSIP


The two Lituus roots in helix form:


### 3.45 Helix Derivatives, Lituus

An example of helix derivatives in 3Di using the Lituus and only the positive root:

$$
y+i z=\frac{d^{n}}{d x^{n}}\left(x^{-\frac{1}{2}} e^{f i x}\right)
$$

The helixes rotate with each derivative. With frequency, $f=1$, the amplitudes are the same except for the asymptotic part of the graph. With frequencies other than 1 the amplitudes expand exponentially with each derivative.

With $f=1$ :


With $f=1.25$, the original helix:


The first derivative is in violet:


The second derivative is in red:


And, the third derivative is in blue:



### 3.46 Square Cornu Spiral/Fresnel Integrals

The Cornu Spiral ${ }^{(*)}$ and Fresnel Integrals ${ }^{(*)}$


Are just the side and front views of the spiraling helix with decreasing amplitude for equation:

$$
y+i z=\int_{0}^{x} e^{i s^{2}} d s
$$



As we saw in sections 3.1 The Elliptic Helix, and 3.2 The 'Cusped Helix, by adding a second term as a reciprocal, amplitude and frequency characteristics are modified. I.e.:

$$
y+i z=\int_{0}^{x}\left(A e^{a i s^{2}}+B e^{-b i s^{2}}\right) d s
$$

The spiraling helix may be given different geometric shapes.

So, with $a=3, b=1, A=.425, a+b=4$ determines the number of cusps:


And, the side view in blue and the front view in black take the 'Square Cornu' shape:

or, with $a=2, b=1, A=.6, a+b=3$, a 'Triangular Cornu':


Or, with $a=1.2, b=.125, A=B=1.5$ :


See section
12.41 Cornu Spiral/Fresnel Surface for putting a surface on the Cornu Spiral.

### 3.5 Conversion From 2D Polar to 3Di Coordinates

### 3.51 Cochleoid

In case it hasn't been noticed, in switching coordinate systems from 2D polar to 3Di for these spirals, whatever operations are done on theta; if those same operations are done on the coefficient ' $x$ ' in front of the normal helix, the specific spiral shapes will result in RSIP.

To illustrate, the Cochleoid has the polar equation:

$$
r=a \frac{\sin \theta}{\theta} \quad a=1
$$

Beginning with the basic helix equation:

$$
y+i z=e^{i x}
$$

Then, placing the coefficient ' $x$ ':

$$
y+i z=x e^{i x}
$$

Then, performing the same operations on ' $x$ ' as on ' $\theta$ ':

$$
y+i z=\frac{\sin x}{x} e^{i x}
$$

And, with the usual change in coordinates:

2D $x=3 D i y$
$2 D y=3 D i i z$
interval $0 \leq x \leq 9 \pi$ :

## 2D Polar



3Di RSIP


### 3.52 Conchoid

More generally, to convert from 2D polar to 3Di coordinates, there are two steps:

1) Replace $\theta$ by $x$, and $r$ by $r(x)$.
2) Insert $r(x)$ as a coefficient of the helix.

Virtually any curve, no matter how exotic, that is expressed in polar coordinates can be converted to 3Di coordinates.

Here are the 2D projections from the resulting space curve for the Conchoid in 3Di:

2D Polar form:

$$
r=a+b \sec \theta
$$

3Di form:

$$
y+i z=(a+b \sec x) e^{i x}
$$

RSIP with interval $-\pi \leq x \leq \pi$ :


The horizontal line is the graphing software's attempt to connect the graph across $\pm \infty$ at $x=\frac{\pi}{2}, \frac{3 \pi}{2}$

The FRP graph is continuous. Interval $-2 \pi \leq x \leq 2 \pi$ :


TIP with interval $-2 \pi \leq x \leq 2 \pi$ :


The vertical lines are the graphing software connecting $\pm \infty$.

### 3.6 Spira Mirabilis 6 New 3Di Equations for the Equiangular Spiral

Helixes provide an opportunity to generate completely new equations for many of the historical curves.

The Equiangular or Logarithmic Spiral, in addition to the polar to 3Di conversion, has some other interesting ways that it can be generated. For the most part the new forms may be made as exact as desired by adjusting coefficients, and with some, adjusting graphing axes.
i. the historical form

$$
\begin{aligned}
& r=a e^{\theta \cot b} \\
& y+i z=\left(a e^{x \operatorname{cotb}}\right) e^{i x} \\
& y+i z=e^{x(f+g i)} \\
& y+i z=d(\ln c i)^{x} \\
& y+i z=d(e+i)^{i x} \\
& y+i z=d\left(-e^{6 \pi}\right)^{i x} \\
& x^{i^{w}} \text { and } e^{x i^{w}}
\end{aligned}
$$

ii. the 3Di converted form
iii. the 'complex exponential' form
iv. a logarithmic form
v. a complex base form
vi. a negative base form
vii. an 'exponential rotator' $\left(i^{w}\right)$ form

The logarithmic, complex base, and negative base forms generate the equiangular spiral in side view from very different helixes. (See section 6.3 The Helix Base $\beta$ and Wavelength $\lambda$ for helixes with different bases.) So the various coefficients have to be adjusted taking into consideration helix direction - real, imaginary, or both - and whether the helix opens toward positive or negative $x$, as well as helix development, clockwise (CW) or counterclockwise (CCW). See sections below.

### 3.61 In 3Di Converted Form.

Historical equation:

$$
r=a e^{\theta c o t b}
$$

And so:

$$
r(x)=a e^{x c o t b}
$$

And, in 3Di:

$$
y+i z=r(x) e^{i x}=\left(a e^{x \operatorname{cotb}}\right) e^{i x}
$$

Additionally, the $\cot b$ function, as well as any trigonometric function, is equivalent to using the real and imaginary parts of the basic helix function.
E.g.,

$$
\cot b=\frac{\text { reale }^{i b}}{\text { image }^{i b}}
$$

with $a=1$ and $b=\frac{7 \pi}{16}$, interval $-12 \pi \leq \theta, x \leq 12 \pi$ :

2D Polar


3Di RSIP


Below, the FRP (front in violet) and TIP (top in red) views show spiraling asymptotic amplitude to the horizontal axis in the negative ' $x$ ' direction, and increasing without limit amplitude in the positive ' $x$ ' direction. They also show a fixed wavelength of $2 \pi$. Changes in $b$ only effect the amplitude, not the wavelength:


The three orthogonal plane projection views together:


And, the spiraling helix itself:


### 3.62 The 'Complex Exponential' Form

$$
y+i z=e^{x(f+g i)}
$$

Here, this helix is correlated to the standard form by:

$$
\begin{array}{ll}
\text { standard form } & r=a e^{\theta \operatorname{cotb}} \\
\text { this exponential form } & y+i z=e^{x(f+g i)}
\end{array}
$$

The spirals will correlate with properly chosen coefficients, e.g.,:

$$
\begin{array}{ll}
\text { Example 1: } & b=1.29085, f=1, \quad g=3.47614 \\
\text { Example 2: } & b=1.2225, \quad f=1, \quad g=2.754070 \\
\text { Example 3: } & b=.689 . ., f=2, \quad g=1.647
\end{array}
$$

And switching the horizontal and vertical axes in RSIP is necessary for the helix form.

## Example 1:



## Example 2:



Example 3:


See section 11.21 Spira Mirabilis Again, for more on the spiral's relationship to rotating exponentials.

### 3.63 In Logarithmic Form

The Equiangular Spiral is also called the 'Logarithmic Spiral'; and it is striking to note that this spiral can be generated by using the logarithm of an imaginary number as a base!

The equation:

$$
y+i z=i^{x}
$$

generates a helix with constant amplitude 1, just like the 'Euler Helix' (see section 6.0 Euler's Formula Upgraded, Helix and Spiral Functions), except that this helix has a wavelength of 4, rather than $2 \pi$. Further, by inserting a coefficient to $i$ :

$$
y+i z=(C i)^{x}
$$

various spiraling helixes are generated and with $C=\frac{\pi}{2}$ :

$$
y+i z=\left(\frac{\pi i}{2}\right)^{x}
$$

which is:

$$
y+i z=(\ln i)^{x}
$$

By inserting coefficients $c$ and $d$ :

$$
y+i z=d(\ln c i)^{x}
$$

This may be made as exact as desired by adjusting $a$ and $b$ in the converted form, with $d$ and $c$ in the logarithmic form.

## Examples:

i. For $a=1, d=1$ and $c=1, b=1.290859$.
ii. For $a=1, d=1$ and $c=2, b=1.133151$ to seven significant digits.
iii. For $a=2, d \cong-.8106$ and $c=1, b=1.290859$.

Here are the three 2D projection graphs superimposed on one another for the $\ln c i$ base helix.

Note wavelength $=4$.

Violet $=$ FRP
Red = TIP
Blue $=$ RSIP

Interval $-12 \pi \leq x \leq 12 \pi$ :


The same three 2D projection graphs orthogonal in space:


And superimposing the helix itself, in black, the asymptotic amplitude is in the negative ' $x$ ' direction:


### 3.64 In Complex Base Form

The complex base form generates an Equiangular Spiraling Helix with the following equation:

$$
y+i z=d(e+i)^{i x}
$$



This differs from the converted/historical form in the following ways:

1) The spiraling asymptotic amplitude is in the positive ' $x$ ' direction rather than the negative ' $x$ ' direction;
2) As we stand and face the positive ' $x$ ' direction the development is CW rather than CCW;
3) The fixed wavelength is approximately 5.90826 rather than $2 \pi$;

To offset the differences it is necessary to switch the graphing axes for one or the other. Then, this form, as the coefficient $d$ is changed, lines up with the converted/historical form in side view in a periodic way! But only for certain values of $a$ and $b$. Since there is only one coefficient, $d$ in this form, it may be possible to insert another coefficient (similar to $c$ in the logarithmic form) that lines them up for other values of $a$ and $b$.

So for example, with $a=1, b=1.25075$, the spiral lines up for:
$d=-.0740,-.00918,-.00114, .00325$ and others.

Converted/historical form in blue, complex base form in red:


Animation 11 'Equiangular Spiral Complex Base Form'

### 3.65 In Negative Real Base Form

When using a negative real base like $\beta=-e$ the two dimensional wavelength is $2 \pi$, but the amplitudes become so large and so small so quickly, that it is difficult to see exactly what is happening.

For example with:

$$
y+i z=(-e)^{i x}
$$

the FRP graph is:


And, the three dimensional graph is:


The larger the base becomes, the shorter is the 2D wavelength. So, if the base is made extremely large:

$$
\begin{gathered}
\beta=e^{6 \pi}=1.53553 \times 10^{8} \\
y+i z=(-\beta)^{i x}
\end{gathered}
$$

then, the 2 D wavelength $=\frac{1}{3}$ :


And, the graphs become somewhat more manageable, especially in the three-dimensional view:



## Animation 12 'Equiangular Spiral Negative Base Form'

As with the complex base form, it is necessary to switch 2D graphing axes in one or the other base forms. And then, by adding the adjusting coefficient as in the other forms:

$$
y+i z=d(-\beta)^{i x}
$$

this spiral will line up with the converted/historical form similarly to the complex base form: i.e., periodically for certain values of $a$ and $b$. And, like the complex form it may be possible to insert another coefficient, $c$, somewhere that lines them up for other values of $a$ and $b$.
E.g., with $a=1, b=1.40556, d=-.0334365, .457446$, and others.

See section 6.0 Euler's Formula Upgraded, Helix and Spiral Functions for a more general analysis of helixes and spirals and an expanded interpretation of Euler's identity.

### 3.66 An 'exponential rotator' $\left\lceil i^{w}\right\rceil$ form:

$$
y+i z=x^{i^{w}} \text { and } y+i z=e^{x i^{w}}
$$

See section 9.40 The Equiangular Spiral and Cardioid Motion

### 3.7 Additional Terms in the Helix Equation

The equation for the elliptical helix (section 3.1 The Elliptic Helix):

$$
y+i z=A e^{i x}+B\left(e^{i x}\right)^{-1}
$$

is a special case of the equation for 'cusped' helixes, circular cycloids and trochoids, rose curves, foliums, etc. (sections 3.2 The 'Cusped Helix to 3.26 Epicycloid, Hypotrochoid, Epitrochoid, Hypocycloid);

In this elliptical helix equation, $a=b=1$ and $A \neq B$

$$
y+i z=A e^{a i x}+B\left(e^{b i x}\right)^{-1}
$$

and then by allowing for different 'frequencies' on each term, this basic idea can be extended to combine any number of helixes:

$$
y+i z=i^{k}\left(A e^{a i x}+B e^{b i x}+C e^{c i x}+D e^{d i t}+\ldots \ldots\right)
$$

The coefficient $i^{k}$ serves as a simple 'rotator'. This is for purposes of matching side views of the resulting helixes to other graphs, or for orienting the side view a certain way (See section 9.39 A Simple Rotator) and so on.

Additionally, any of the amplitude or 'frequency' coefficients can also be functions:
E.g., $\quad A=A(x), \quad a=(a(x))^{\left(a_{2}(x)\right)}, c=(c(x))^{\left(c_{2}(x)\right)^{\left(c_{3}(x)\right)^{\left(c_{4}(x)\right)}}}$

Needless to say, this creates a vast array of unlimited possibilities for new functions, any and all of which may be differentiated. See sections 7.7 Higher Level Exponents.

### 3.71 Cardioid

The historical equation for the Cardioid is:

$$
r=2 a(1+\cos \theta)
$$

with $a=1$. Using this more generalized helix equation, the Cardioid:

$$
y+i z=i^{k}\left(A e^{a i x}+B e^{b i x}+C e^{c i x}+D e^{d i t}\right)
$$

has the following coefficients:

$$
\begin{array}{cccc}
a=2, & b=1, & c=0, & d=0 \\
A=1, & B=2, & C=1, & D=0
\end{array}
$$

Note that, in this case, we did not directly convert the polar form to the helix form, but rather just found the right coefficients, summed two helixes with a constant ,and rotated it. I.e.,

$$
y+i z=i\left(e^{2 i x}+2 e^{i x}+1\right)
$$

2D Polar; interval: $0 \leq \theta=x \leq 2 \pi$

greg ehmka, 2013

As the tracing coordinates show this equation is exact at least to 8 significant digits.

The full Cardioid Helix on the interval; $-3 \pi \leq x \leq 3 \pi$

Animation 13 'Cardioid Helix’

### 3.72 Unification Note

As a side note of interest, in the external link here, the historical form for the Cardioid, in addition to polar coordinates, is also given in a two dimensional Cartesian equation. I.e.,

$$
\left(x^{2}+y^{2}-2 a x\right)^{2}-4 a^{2}\left(x^{2}+y^{2}\right)=0
$$

Given that the helix equation which is a transcendental function, for the Cardioid is:

$$
y+i z=i\left(e^{2 i x}+2 e^{i x}+1\right)
$$

this gives an algebraic form and a transcendental form for the same object in the same coordinates. This may be of significant potential usefulness in the eventual unification of algebraic and transcendental functions.

### 3.73 Cardioid Derivatives

Going to the Cardioid derivatives, with coefficients as in the previous section, the first derivative (RSIP view) of the Cardioid is a Limacon, enlarged and rotated:

$$
y+i z=\frac{d}{d x}\left(i^{k}\left(A e^{a i x}+B e^{b i x}+C e^{c i x}+D e^{d i t}\right)\right)
$$



The second derivative enlarges and rotates further:

$$
y+i z=\frac{d^{2}}{d x^{2}}\left(i^{k}\left(A e^{a i x}+B e^{b i x}+C e^{c i x}+D e^{d i t}\right)\right)
$$


3.74 Freeth's Nephroid with Derivatives

2DPolar:

$$
r=g\left(1+2 \sin \frac{\theta}{2}\right) \quad g=1
$$

3Di with coefficients:

$$
\begin{gathered}
y+i z=i^{k}\left(A e^{a i x}+B e^{b i x}+C e^{c i x}+D e^{d i t}\right) \\
a=3, \quad b=2, \quad c=1, \quad d=0 \\
A=1, \quad B=1, \quad C=1, \quad D=0 \\
k=3
\end{gathered}
$$

2D Polar graph; Interval; $-2 \pi \leq x \leq 2 \pi$


3Di RSIP graph; Interval; $-2 \pi \leq x \leq 2 \pi$ As with the Cardioid this equation is exact.


## Freeth's Nephroid space helix:



Freeth's Nephroid, first derivative in 3Di RSIP view:


Freeth's Nephroid, second derivative in 3Di RSIP view:


### 3.75 Limacon

2D Polar equation:

$$
r=g+2 \text { h } \cos \theta \quad g=h=2
$$

The 3Di helix equation which, as the tracing coordinates show, is exact:

$$
\begin{gathered}
y+i z=i^{k}\left(A e^{a i x}+B e^{b i x}+C e^{c i x}+D e^{d i t}\right) \\
a=4, b=2, c=d=0 \\
A=B=C=2, D=0
\end{gathered}
$$

$$
k=1
$$



3Di RSIP graph;


The three orthogonal planar views in space: Interval: $-4 \pi \leq x \leq 2 \pi$


The Limacon space helix:

### 3.76 Nephroid

2D Polar equation for the Nephroid;

$$
x=a(3 \cos t-\cos 3 t), \quad y=a(3 \sin t-\sin 3 t) \quad a=1
$$

The 3Di helix equation for the Nephroid is exact:

$$
\begin{gathered}
y+i z=i^{k}\left(A e^{a i x}+B e^{b i x}+C e^{c i x}+D e^{d i t}\right) \\
A=B=C=D=1 \\
a=3, b=c=d=1 \\
k=0
\end{gathered}
$$

2D Polar with interval: $0 \leq \theta \leq 2 \pi$


3Di RSIP with interval: $0 \leq x \leq 2 \pi$


$$
A=2 \quad A=3
$$



The three orthogonal planar views in space: $-4 \pi \leq x \leq 2 \pi$


### 3.77 Talbot's Curve with Derivatives

The historical expression for Talbot's Curve is the two parametric equations:

$$
\begin{gathered}
x=s\left(\frac{\left(a^{2}+\left(f^{2} \sin ^{2} t\right)\right)(\cos t)}{a}\right) \\
y=s\left(\frac{\left(a^{2}-2 f^{2}+f^{2} \sin ^{2} t\right)(\sin t)}{b}\right)
\end{gathered}
$$

In 3Di the generalized helix equation is:

$$
y+i z=i^{k}\left(A e^{a i x}+B e^{b i x}+C e^{c i x}+D e^{d i t}\right)
$$

and an approximation for Talbot's Curve has coefficients:

$$
\begin{array}{rccc}
a=3, & b=-1, \quad c=-3, & d=1 \\
A=.6, & B=3.4, \quad C=-2, & D=5 \\
& k=1 &
\end{array}
$$

For the historical equations to make a general match the coefficients used are:

$$
s=7(\text { scaling factor }), \quad a=1, \quad b=.5, \quad f=.925
$$

Historical curve in violet;
3Di RSIP in black:


And a side view of the three helix space curves with:

The 3Di Talbot's curve in black;
First derivative in blue;
Second derivative in red:


Talbot's curve with derivatives

### 3.78 Cowboy Hat with Derivatives

Providing a little comic relief, and to show some of the endless possibilities of a generalized helix equation with only four terms:

$$
y+i z=i^{k}\left(A e^{a i x}+B e^{b i x}+C e^{c i x}+D e^{d i t}\right)
$$

The coefficients for a "Cowboy Hat Helix" are:

$$
\begin{array}{ccc}
a=-3, & b=-1, \quad c=2, \quad d=0 \\
A=1, & B=4, \quad C=2, \quad D=5 \\
& k=0
\end{array}
$$

3Di RSIP; Interval: $0 \leq x \leq 2 \pi$


Black = Space helix for original equation;
Blue = first derivative;
Red = second derivative:


Animation 14 'Cowboy Hat with Derivatives'

### 3.8 The Serpentine, Witch and a "Circle"

Referring back to section 2.6 Conics in 3Di , the hyperbola generated by the conic:

$$
\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=C
$$

has an orthogonal ellipse in the center of it that joins the two vertices of the hyperbola, and in fact, produces two space functions with 'bifurcation points' where the graph makes an abrupt turn off the FRP onto the TIP:


We can contrast this type of 'conic hyperbola' with the family of hyperbolas given by:

$$
x y= \pm C
$$



As well as not being continuous at zero, these hyperbolas do not have orthogonal ellipses in between the vertices due to there being no imaginary numbers generated in the solutions.

Setting C = 1 the graph of:

$$
x y=1
$$



As is normal, this hyperbola exists exactly on the $x y$ plane, does not have a circle in the middle and is not continuous at zero. If we were to now add an imaginary constant term to the input:

$$
(x+b i) y=1
$$

Or:

$$
y+i z=\frac{1}{x+b i}
$$

With $b=-1$ :

The FRP in violet gives the Serpentine.
The TIP in red gives the Witch of Agnesi.
The SIP in blue gives a circle that converges to zero as $x$ goes $\pm \infty$.


The three plane projections orthogonal in space:


And the full 3Di graph:


Animation 15 'Serpentine, Witch and Circle'

The historical equation for the Witch of Agnessi is:

$$
y\left(\mathrm{x}^{2}+a^{2}\right)=a^{3}
$$

Or:
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$$
y=\frac{a^{3}}{\left(\mathrm{x}^{2}+a^{2}\right)}
$$

And, the historical equation for the Serpentine is:

$$
y=\frac{a^{2} x}{\left(x^{2}+a b\right)} \quad a b>0
$$

With $a=b=1$ these become respectively:

$$
y=\frac{1}{\left(x^{2}+1\right)} \quad \text { and } y=\frac{x}{\left(x^{2}+1\right)}
$$

In three dimensions both of these graphs show up only on the front real plane (FRP) as follows:

Serpentine in violet:
With of Agnessi in red:


If we apply the imaginary coefficient $i$ to the Witch (red), the effect is to make the two graphs orthogonal to one another, meaning to rotate the Witch graph a quarter turn, $\frac{\pi i}{2}$, in the imaginary direction. Then the dependant variable values will appear on the $i z$ axis rather than the $y$ axis.

With the graphs are orthogonal they appear as follows:


Algebraically, we have transformed the Witch graph from:

$$
y=\frac{1}{\left(x^{2}+1\right)}
$$

to:

$$
i z=\frac{i}{\left(\mathrm{x}^{2}+1\right)}
$$

Keeping the Serpentine graph the same, we now have:

$$
y+i z=\frac{x}{\left(x^{2}+1\right)}+\frac{i}{\left(\mathrm{x}^{2}+1\right)}
$$

In this form, the two curves together are identical to the 3Di equation:

$$
y+i z=\frac{1}{x-i}
$$

And so, we have an identity:

$$
\frac{1}{x-i} \equiv \frac{x}{\left(x^{2}+1\right)}+\frac{i}{\left(\mathrm{x}^{2}+1\right)} \equiv \frac{x+i}{\left(x^{2}+1\right)}
$$

In normal algebraic manipulation this is just multiplying numerator and denominator by $(x+i)$. But in 3Di coordinates this algebraic manipulation can have potentially deeper significance in combining plane curves into space curves or the reverse.

For alternative values of $a$ and $b$, the identity has yet to be worked out.

### 3.9 Additional Historical Curves

## Semi Cubicle Parabola; See section 4.5 Cusps;

Newton's Diverging Parabolas; See section 4.55 Elliptic Curve Bifurcation

Lame Curves; See section

### 4.59 Lame Curve Bifurcation

Tschirnhaus' Cubic; See section 4.7 Complex Coefficients - Polynomials in Space

### 4.0 Polynomials in 3Di

### 4.1 Quadratic Input and Output

Under certain conditions the solutions to a $2^{\text {nd }}$ degree polynomial equation give real numbers, and under other conditions the solutions give complex numbers.

For example:

$$
x^{2}+2 x-2=0
$$

will give the real number solutions: $-1 \pm \frac{\sqrt{12}}{2}$
while:

$$
x^{2}+2 x+2=0
$$

gives the complex number solutions: $-1 \pm i$

In the first example, $x^{2}+2 x-2=0$, the graph, meaning the function $x^{2}+2 x-2=y$ looks like:


And, it is easy to see that the two roots are real and equal to: $-1 \pm \frac{\sqrt{ } 12}{2}$
They are, in fact, located at what are usually called 'the zeros'. And, this corresponds with the original equation being set to zero and having two solutions or roots.

The second equation: $x^{2}+2 x+2=0$, meaning the function: $x^{2}+2 x+2=y$, looks like:


The graph itself does not intersect the x -axis, and so 'the zeros', $-1 \pm i$, do not fall on the x axis and do not fall on the real plane at all.

In 3Di we know exactly where these roots do fall. Notice the red dot:


The red dot represents the two roots that are located at: $(\mathrm{x}, \mathrm{y}, \pm \mathrm{iz})=(-1,0, \pm \mathrm{i})$ They are not on the real plane and are 'the zeros' of the equation: $x^{2}+2 x+2=0$

Both roots show up if we look at the TIP:


Polynomial equations such as these can be a little confusing due to the fact that when solving for $x, x$ represents the solution(s) of the equation, rather than input to a function. So, in the function :

$$
x^{2}+2 x=y
$$

the input is $x$ and the out put is $y$, but in the polynomial equation:

$$
x^{2}+2 x+2=0
$$

$x$ is the solution and input and output, as such, do not strictly apply. But, this is actually somewhat untrue, and grounds for further confusion, because the equation:

$$
x^{2}+2 x+2=0
$$

is identical to the function:

$$
x^{2}+2 x=y \quad \text { IF } y=-2
$$

If we apply the quadratic equation to the above polynomial:

$$
x=\frac{-b \pm \sqrt{ }\left(b^{2}-4 a c\right)}{2 a} \quad a=1, b=2, c=2
$$

becomes:

$$
x=\frac{-2 \pm \sqrt{\left(2^{2}-4(1)(2)\right)}}{2}
$$

Now here is the interesting part: if we, similarly, apply the quadratic to the whole function, we have:

$$
x=\frac{-2 \pm \sqrt{\left(2^{2}-4(1)(y)\right)}}{2}
$$

And then, if we find the solutions, $x$, for each value of $y$, we will be building the threedimensional parabola backwards from $y$ to $x$, rather than forwards from $x$ to $y$, and the notions of input and output have actually been reversed!

This means that $y$ is the input which specifies where the 'solutions' are located for that value of $y$. Consequently, $x$ is, in this sense, the output! Graphing the parabola in this way allows us to include the portion of the graph that is below the vertex which lies in the third dimension.

As an example, if we were to graph 'the roots' of some successive polynomial equations that differ only by a constant, for example:

$$
x^{2}+2 x+C=0 \text { with } C=2,3,4
$$

this would be the same functionally as $y$ being the input and $x$ being the output, as in:

$$
x^{2}+2 x=y \text { with } ; y=-2,-3,-4
$$

In the 3Di top view, this would look like the following:
(Colors denote pairs of solutions.)
$y=-2$ in red,
$y=-3$ in blue,
$y=-4$ in black.


In FRP these would look like the three graphs:


If we extend this idea to allow $y$ to take on all values, then the three-dimensional graph includes both the portion above the vertex and below the vertex, and looks like:


Animation 16 'Quadratic Bifurcation'

### 4.11 Quadratic Nonlinearity

From the above, we can see that in the usual solving of a polynomial equation when we find the roots, which is equivalent to finding the zeros, - we are in actuality sliding the, would be, parabola (were we to graph one) up and down in accordance with the constant C , since $y$ must always be zero. In so doing, we often find complex numbers.

But, in graphing the function itself there are no complex numbers generated, and so the graph, seemingly, "does not exist" below the vertex and so only lies in two dimensions on the real plane.

When we reverse the roles of $x$ and $y$ by using the quadratic equation in this threedimensional way, if complex numbers are generated, then the imaginary part, along with $x$, becomes part of the output and will lie on the $i z$ axis. This gives us:

$$
x+i z=\frac{-b \pm \sqrt{ }\left(b^{2}-4 a(c-y)\right)}{2 a}
$$

which, as said, reverses the input and output roles of $x$ and $y$ in the usual functional equation:

$$
a x^{2}+b x+c=y
$$

and this provides the imaginary numbers for a three-dimensional graph.

The quadratic equation, of course, gives two solutions and so, just as in section 2.61 Conic Nonlinearity, we have the two functions:

$$
\begin{aligned}
& x+i z=\frac{-b+\sqrt{ }\left(b^{2}-4 a(c-y)\right)}{2 a} \\
& x+i z=\frac{-b-\sqrt{ }\left(b^{2}-4 a(c-y)\right)}{2 a}
\end{aligned}
$$

The first is in red and the second in black:


### 4.2 Imaginary Polynomial Coefficients

In 3Di, since we know how to graph the final result, we can use the imaginary coefficient $i$ pretty much anywhere in the equation and still generate a graph that makes sense.

### 4.21 Imaginary Trans/ation

Beginning with the usual equation for an FRP Hyperbola/TIP Ellipse:

$$
x^{2}-y^{2}=1
$$

and then, adding a translation term to what will be the dependant variable - meaning, we will be solving for ' $y$ ':

$$
\begin{gathered}
x^{2}-(y+i)^{2}=1 \\
(y+i)^{2}=x^{2}-1 \\
y+i= \pm\left(x^{2}-1\right)^{\frac{1}{2}} \\
y=\left[ \pm\left(x^{2}-1\right)^{\frac{1}{2}}\right]-i
\end{gathered}
$$

At this point, $y$ itself becomes complex. So proceeding to 3 Di , with the real and imaginary parts of $y$ being graphed on the vertical and depth axes respectively:

$$
y+i z=\left[ \pm\left(x^{2}-1\right)^{\frac{1}{2}}\right]-i
$$

Untranslated:


Translated in the $-i$ direction:


The additional term, modifying $y$, can be used to translate either vertically along $y$ (real), or along $i z$ (imaginary), or both.

Another example:

$$
x^{2}+2 x+4=y \quad \text { in black }
$$

with an added imaginary coefficient on the constant term:

$$
x^{2}+2 x+4 i=y+i z \text { in blue }
$$

will produce an 'imaginary translation' off of the real plane, directly forward along the 'depth axis', and downward:


### 4.22 Imaginary Rotation and Concavity

Placing an imaginary coefficient on the linear term:
$x^{2}+2 i x+4=y$ (in red)
translates and rotates the graph, plus it softens the concavity:


### 4.23 Imaginary 'Tilt'

And by placing the $i$ on the first term:
$i x^{2}+2 x+4=y$ (in red)
the graph is translated, rotated and tilted:

### 4.3 Helixes as Polynomial Coefficients

## Example One

In addition to imaginary or complex coefficients in a polynomial, helixes can be used as coefficients also. Or, the other way around - meaning, polynomials as coefficients to the helixes.

In this context, consider again the Spiral of Archimedes in section 3.41 Archimedes Spiral (with larger frequency $f$ for visibility.)

$$
y+i z=x e^{f i x} f=3
$$

Showing only half of the spiral - meaning, interval: $0 \leq x \leq 4 \pi$

with a second degree exponent on $x$ :

$$
y+i z=x^{2} e^{f i x}
$$



## Example Two

If a helix is used as a coefficient on the linear term of a quadratic:

$$
y+i z=x^{2}+x e^{i x}-8
$$

The 3Di graph:


The RSIP graph:


The FRP real values in violet.
The TIP imaginary values in red.


## Example Three

Also, we see that in 3Di, the coefficients of a quadratic, or any polynomial, may also be variables as well as real or complex constants. And, the resultant object will still make sense and be able to be graphed.

For example, here is an interesting but unusual looking quadratic. It serves as a reminder of the difference between the FRP projection and the FRP actual. The curve has two distinct branches, only certain points of which lie on the actual FRP, as opposed to the FRP projection. Those points are where $x= \pm \pi$ and multiples thereof.

$$
x^{2}-y x+e^{i y}=0
$$

For purposes of using the quadratic equation:

$$
\begin{gathered}
a=1 \\
b=-y \\
c=e^{i y}
\end{gathered}
$$

we apply the quadratic equation, have $y$ as the input, and $x+i z$ as the output, and keep the horizontal, vertical and depth axes the same. Then we have:

$$
x+i z=\frac{-(-y) \pm \sqrt{ }\left(y^{2}-4 e^{i y}\right)}{2}
$$

The three projections for the positive root:
FRP in violet.
TIP in red.
RSIP in blue.


Then, both roots:


Then, in three dimensions:

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## Jump Discontinuity

In the original equation:

$$
x^{2}-y x+e^{i y}=0
$$

after applying the quadratic equation, the dependant variable becomes $x$, which having had an exponent of 2 , gives another example of nonlinearity.

So, the two roots give the two graphs (positive root in red, negative root in black) with a jump discontinuity at $x=0$. The jump discontinuity shows how the two different roots contribute different portions to the two seemingly continuous branches.


### 4.4 A Quadratic Ellipse-Hyperbola in Space

As we saw in section 2.6 Conics in 3Di, the usual conic equation of a hyperbola:

$$
\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=C
$$

actually includes an ellipse in top view, while the usual equation of an ellipse:

$$
\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=C
$$

includes a hyperbola in top view. Therefore, the usual equations of an ellipse and a hyperbola are of the same mathematical object, and this object exists in three dimensions. The $\pm$ operation sign only determines which of the two will show up in the front view. They are in effect quarter-turn rotations of one single three-dimensional object.

At the two points (which are the vertices), where the ellipse and the hyperbola each become the other, a very interesting thing happens. In three dimensions the graph takes an abrupt turn onto the orthogonal plane. If it is curving on the FRP. then at the vertex, it abruptly turns onto the TIP.

Seeing only a hyperbola on the FRP, for example:


The vertices are 'bifurcation points' (See section 4.6 Polynomial Bifurcation) where the graph appears to be splitting into two branches. As we saw in section 2.61 Conic Nonlinearity, this is
not the case, but that two functions occur as a result of taking the two square roots. So, we have the two functions:

$$
\begin{aligned}
& y+i z=+\sqrt{x^{2}-1} \quad \text { (in red), } \\
& y+i z=-\sqrt{x^{2}-1} \quad \text { (in black). }
\end{aligned}
$$



The same kind of 'polynomial bifurcation' (section 4.6 Polynomial Bifurcation ) occurs at the vertex of a parabola, which we saw in section 4.11 Quadratic Nonlinearity. This bifurcation point, among other characteristics, is where two separate functions (e.g., the 'roots graphs' ) of a quadratic meet.

In section 4.6 Polynomial Bifurcation, we will see that this same appearance of polynomial bifurcation occurs in $3^{\text {rd }}, 4^{\text {th }}$ and higher degree polynomials at the local maxima and minima.

In addition to the above 3D ellipse-hyperbola object using a conic equation, we can also use quadratics to generate the ellipse-hyperbola object.

In two dimensions, if we graph something we can call a 'reciprocal sum' (violet), or a 'reciprocal difference' (blue), with $x$ as input on the horizontal axis and $y$ as output on the vertical axis, we have the following:

$$
y=x+\frac{1}{x}, \quad y=x-\frac{1}{x}
$$



And then, if we put one of them - say, the reciprocal sum - into quadratic form:

$$
x^{2}-x y+1=0
$$

and then, apply the quadratic equation:

$$
x=\frac{-(-y) \pm \sqrt{ }\left(y^{2}-4\right)}{2}
$$

some interesting things occur: First, as before, when taking roots, there will be two functions generated. And second, by allowing $y$ to take on all values, complex numbers are likely to be generated, which will give us a three dimensional graph. And so, we have the following two functions:

$$
\begin{aligned}
& x+i z=\frac{-(-y)+\sqrt{ }\left(y^{2}-4\right)}{2} \\
& x+i z=\frac{-(-y)-\sqrt{ }\left(y^{2}-4\right)}{2}
\end{aligned}
$$

In the original reciprocal sum function (in two dimensions), the vertices are just part of the smoothness of the graph. But, in the pair of functions that occurs after applying the quadratic equation, they will be the 'bifurcation points' at which multiple functions meet.

Here, we are graphing $y$ as input, but keeping it on the vertical axis, and then, $x$ is the output on the horizontal axis. The $i z$-axis becomes the third-dimension depth.

First, we have the real component of the positive root projected to two dimensions - the FRP. Notice there is a slope to the straight line connecting the vertices. This shows that the orthogonal ellipse does not lie on the TIP, but in space.

Interval: $-12 \leq x \leq 12$


And then, we have the imaginary component of the positive root projected to the TIP. Notice the graph continues along the horizontal axis outside the space ellipse. This shows that the hyperbola portion lies on the FRP.


The positive root in three dimensions:


The negative root: real component in blue, imaginary component in red, three dimensions in black:




The two separate functions, positive root in black, negative root in grey:


And, both roots showing the space ellipse-hyperbola generated from a quadratic:


Depending on the usual quadratic coefficients, $a, b, c$ (which may be real, imaginary or complex), the attributes like eccentricity, orientation and size of the object may be altered. And, as we saw previously, the object may be rotated, tilted and even translated such that no part of it lies on the FRP.

The point here is that all, part, or none of the object may show up in one of the normal planes, and/or only be in space somewhere. Then, it must be projected to a plane; and that may have both a useful and a distorting effect that has to be considered - including the differences between the FRP projected and the FRP actual.

### 4.5 Cusps

In addition to vertices in two dimensions being indicators of bifurcation points, cusps can indicate bifurcation points as well.

### 4.51 Helix Cusp

In the cases where we reverse the function - meaning graphing $x$ in terms of $y$, rather than graphing $y$ in terms of $x$ - the vertices, in addition to being where two functions meet, are also the places where the graph makes an abrupt turn, usually onto or off of one of the usual planes. Cusps can show this also even in the original function. They are often a place where these abrupt turns, onto or off of a plane, take place. In the next example, a helix abruptly turns onto the FRP. It uses an $i-$ base helix with a second level exponent.

In the $i$-base helix equation:

$$
y+i z=i^{(a x)^{\frac{1}{2}}} a=3
$$

The curve is a helix on the positive $x$ side, and an asymptotic curve, turning onto the FRP and remaining on the FRP for negative $x$ :


The FRP shows this cusp at $(x, y)=(0,1)$. Then the real values become asymptotic to the horizontal axis:


The TIP shows that the imaginary part of the output falls to zero, at $x=0$, from positive $x$, and remains at zero:


And, the RSIP shows the constant amplitude helix transitioning to the FRP. Note the continuation of the blue line graph directly on the vertical axis:


### 4.52 Polynomial Cusp

This equation again shows an abrupt turn onto the FRP:

$$
y+i z=\left(x^{3}-2 x^{2}+x-4\right)^{\frac{1}{4}}
$$

The graph in space comes in from negative $x$, with real and imaginary components equal to one another until $x=2.3145962$, at which point imaginary values become zero, and remain at zero, as real values increase again, while remaining on the FRP:


The FRP shows the real values decreasing to $x=2.3145962$, and then increasing again. (The graphing software has a little difficulty with this one: not quite making it to zero.)


In the TIP, the imaginary values go to zero and remain there:


The RSIP is a nice, straight line showing that the real and imaginary components are equal all the way to zero. Then, at zero, there would be a turn onto the FRP, which would continue as a vertical blue line directly on the axis. (Only a partial interval is pictured as the graphing software has difficulty making the turn.)


### 4.53 Semicubical Parabola

The cusp of the Semicubical Parabola is also a turn (more of a gradual one) onto the TIP. By allowing $x$ to take on negative values in the equation:

$$
y^{2}=x^{3}
$$

we get the two functions:

$$
\begin{gathered}
y+i z=\left(x^{3}\right)^{\frac{1}{2}} \\
y+i z=-\left(x^{3}\right)^{\frac{1}{2}}
\end{gathered}
$$

When graphed together in three dimensions, the usual graph is seen for positive $x$; but, for negative $x$, the graph transitions onto the TIP. The imaginary values that are generated create two branches in top view - directly on the TIP, since the real values for negative $x$ are all zero.


The FRP shows the real values which are the usual graph for positive $x$. The upper half is the positive root, and the lower half is the negative root. Both have real value zero for all of negative $x$. Note the continuing blue line along the x -axis.


Notice that in TIP, the imaginary values are zero for positive $x$, just as in FRP, the real values are zero for negative $x$.


Since both parts of the graph lie directly on the FRP and TIP planes respectively, the RSIP shows the graph only on the two axes in blue.


And then, remembering that there are two functions, each generating half of the graph - but, each half contains both real and imaginary parts, positive root in black, negative root in red:


## Animation 17 'Semicubical Parabola'

### 4.54 Mordell Bifurcation, Moving the Cusp

A Mordell Curve is a Semicubical Parabola with an additional constant. Once again, the cusp is a transition point from plane to plane (in this case, from on the FRP, directly onto the TIP). But, what makes this curve particularly interesting is that, as the constant changes, it is the cusp the bifurcation itself - that is moving!

The usual equation is:

$$
y^{2}=x^{3}-C
$$

And so, in three dimensions with 3Di, there are the two functions:

$$
y+i z= \pm\left(x^{3}-C\right)^{\frac{1}{2}}
$$

Positive root function in black;
Negative root function in red.

The animation begins with $C=3$, and then to $C=5$, and then down to $C=-5$, and back again. (Once again, the graphing software has a little trouble with the cusp.)

Notice that at $C=0$, the 'half loop' portion of the graph transitions from the TIP onto the FRP, as $C$ goes from plus to minus; then, it transitions back again, as $C$ goes from minus to plus. The bifurcation point itself bifurcates!


Animation 18 'Mordell Moving Cusp'

### 4.55 Elliptic Curve Bifurcation

The usual graphs that we see of Elliptic Curves sometimes have breaks in them: for example, here, where the curve appears to be in two completely separate parts. This is because those graphs only show the FRP. The breaks in the graph are where the graph has transitioned off the FRP and onto the TIP, or the other way around.

Elliptic Curves are Mordell Curves with an added linear term:

$$
y^{2}=x^{3}+B x+C
$$

And so, the two 3Di functions are:

$$
y+i z= \pm\left(x^{3}+B x+C\right)^{\frac{1}{2}}
$$

As before:
positive root function in black, negative root function in red,
with: $B=-3, C=-.5$

Notice the graph appears to be in four segments, and that the two functions meet at the three vertices or cusps. The two functions are on the same plane together in each segment and cross over one another at the vertices. (For a better view, this graph is oriented positive horizontal to the left.)


In two dimensions below are the FRP real values with each of the two functions (red and black) contributing the real segments.


In two dimensions again, overlaying the TIP view with the FRP, the blue and aqua functions contribute the imaginary segments that are orthogonal to the red and black functions which contribute the real segments.


### 4.56 Elliptic Curve Nonlinearity

As defined previously, nonlinearity here means exponents $n$ other than one on the dependent variable:

$$
y^{n}=x^{3}+B x+C
$$

So in 3Di coordinates with, for example, $n=5$ gives the five roots functions:

$$
y+i z=e^{\frac{2 \pi i k}{5}}\left(x^{3}+B x+C\right)^{\frac{1}{5}} \quad k=1,2,3,4,5
$$

The five Demoivre numbers would be the five fifth roots of unity. In decimal and TIP graph form they are:
$n=1, .309017+.95106 i$ in violet
$n=2,-0.809017+.587785 i$ in red
$n=3,-0.809017-.587785 i$ in blue
$n=4, .309017-.951057$ in aqua
$n=5,1$ in black


And the five roots graphs in three dimensions:


### 4.57 Skewed Elliptic Curves

Skewed elliptic curves represent an asymmetric variation of elliptic curves wherein the two functions are not mirror images of one another. Additionally the line segment which connects the axes of the graph does not lie on the $x$-axis but has a slope.

This can be generated by using the concept of nonlinearity and extending it to a quadratic form for the dependent variable. (It can also be extended to a cubic or quartic form.)

Using an equation like:

$$
\left(y^{2}+a y-c x y\right)=\left(x^{3}+b x^{2}+d x+f\right)
$$

and, applying the quadratic equation for $y$ :

$$
A y^{2}+B y+C=0
$$

with:

$$
\begin{gathered}
A=1 \\
B=(a-c x) \\
C=-\left(x^{3}+b x^{2}+d x+f\right)
\end{gathered}
$$

and coefficients:

$$
a=b=2, \quad c=1, \quad d=-1, \quad f=-2
$$

will move the "loops" off the usual planes and out into space, plus the graph is given a slope.


Changing the coefficient $c$ alters the 'skew' of the curve. The two dimensional real plane graph is on the left and the three dimensional graph is on the right for some values of $c$. The 3D graph is oriented positive to the left to compensate for the graphing software's difficulties at the cusps:
$c=1$ (the above example)



With $c=2$ :


With $c=-2$ :


### 4.58 Imaginary Hyperelliptic Curves

Whereas elliptic curves are of degree three for the independent variable and of degree two for the dependent variable, hyperelliptic curves are of degree greater than three for the independent variable. For example:

$$
y+i z= \pm\left(x^{5}-2 x^{4}-7 x^{3}+8 x^{2}+12 x-2\right)^{\frac{1}{2}}
$$

Positive root in red, negative root in blue, positive $x$ to the left


Front real plane view:


Top imaginary plane view:


Here is another example using a polynomial of degree six for the independent variable:

$$
y+i z= \pm\left(x^{6}+1.25 x^{5}+2.625 x^{4}+9 x^{3}-2 x^{2}-9.75 x+3\right)^{\frac{1}{2}}
$$



## Front real plane view:



Top imaginary plane view:


### 4.59 Lame Curve Bifurcation

'Polynomial Non-linearity' means multiple roots of $y$, which generates both complex numbers as well as additional lines (functions) to the graph.

Defining a type of polynomial non-linearity in this way allows each root to be its own independent function with its own independent 3Di space graph.

Again, as in section 2.61 Conic Nonlinearity:

$$
\begin{aligned}
& y^{2}=x^{2}-C \\
& y^{b}=x^{a}-C
\end{aligned}
$$

with $C=4, \quad a=2$, and $b=2$ (square roots),

but, with $a=4$ :

and, with $a=3$, which, in TIP, generates Mordell curves and shapes:


As $a$ increases, the FRP graph comes closer and closer to vertical lines, while the TIP graph generates the historical curves known as 'Lame Curves'.

TIP with: $a=6$


Even exponents bring the TIP curve closer and closer to a rectangle, whereas odd exponents, such as with the Mordell Curves, bring the curve closer and closer to an interesting combination of rectangle and a 'rectangular hyperbola.' Intermediate exponents disclose intermediate shapes.

TIP with: $a=5$


TIP with: $a=5.9$


$$
a=21:
$$



With an odd exponent like $a=21$, and square roots, the two roots graphs have the bifurcation points such that the hyperbolic segments are on orthogonal planes.


While even exponents, $a=20$, and square roots, give the two roots graphs with the "hyperbolic" segments on the same plane.


This is the case with odd and even integer values for $a$. In three dimensions, as $a$ is varied between integer values, the branches of the negative $x$ 'Lame hyperbola' whirl and spin, up and down and back and forth, across the FRP. They effectively rotate in accordance with the changing imaginary values of $i z$. See section 9.31 The Polynomial Morphing Function which also whirls and spins around the $x$-axis in accordance with a changing exponent.


[^2]
### 4.6 Polynomial Bifurcation

Just as with the quadratic equation applied to $2^{\text {nd }}$ degree polynomials (section 4.11 Quadratic Nonlinearity), the procedures for finding roots of cubic and quartic equations will also generate examples of bifurcation, non-linearity and separate roots graphs.

Analogous to quadratic non-linearity, the point in the polynomial roots graph of cubics and quartics where there is an abrupt turn onto a different plane, occurs at what are usually called the 'turning points' - the maximums and minimums. Additionally, these turning points, bifurcation points, vertices, etc. are the points at which two or more functions (roots graphs) meet.

### 4.61 Roots of a Cubic

A fairly typical cubic equation like:

$$
x^{3}+2 x^{2}-x+3=0
$$

will have the three roots: $-2.757279, .3786394 \pm$ i. 9719369
which can be seen in TIP, in red, in the following way.


These points for the roots will show up like this by first solving the equation in one of the usual ways, and then graphing $y$, on the vertical as the input (in this case zero) , and graphing $x+i z$
as the output on the horizontal and depth axes respectively. This is the same procedure that we used in section 4.1 Quadratic Input and Output.

In the FRP these three points will be the three zeros of the graph of equation:

$$
x^{3}+2 x^{2}-x+3=y
$$

Only one of the three points (red) is actually on the FRP. The other two (violet) are projected to the FRP, and so only the real component can be seen:


Ordinarily, this equation has $x$ as input, $y$ as output, and has a two dimensional graph exactly on the FRP. If the equation is slightly rearranged:

$$
x^{3}+2 x^{2}-x+(3-y)=0
$$

and, the Cubic Formula ${ }^{(1)(2)(3)}$ is applied, varying $y$ for as large an interval as we desire (in this case $-15 \leq y \leq 15$ ), and then also continuing to graph $y$ on the vertical, as above, along with $x$ and $i z$, as above, then a remarkable graph results. The three roots obtained with the cubic formula for each $y$ as input give three separate functions. When they are all graphed the following picture results:


Animation 20 'Cubic Polynomial Bifurcation'

Note first the three separate functions; $e^{2 \pi i \frac{1}{3}}$ in blue, $e^{2 \pi i \frac{2}{3}}$ in red, and $e^{2 \pi i \frac{3}{3}}$ in black.

Then notice that the same exact graph that is produced in two dimensions on the FRP is also produced exactly on the FRP in three dimensions; but, each of the three separate functions provides only a segment of the FRP graph. Notice also that the blue and the black graphs do not meet each other, whereas each of them meet the red graph. These meeting points are the polynomial bifurcation points, where the individual graphs make their abrupt qualitative changes in addition to encountering one another.

When using the various cubic formula approaches, since there are two square roots for each cube root, there are six separate values generated in this way; and various choices are generally made as to which ones may be discarded along the way as duplicates. While it is true that taken all together, the points on the graphs of the three branches will be the same regardless of the discarded values, in terms of continuity, they are not graphed in exactly the same way. Different choices of values along the way will contribute different continuous segments to the branches. So, it would appear (a more rigorous analysis would be necessary) that there are six different potential functions, even though they will perfectly overlay so as to generate only three distinct lines.

For contrast, here is another set of colored segments that more generally uses the negative square roots in contrast to the positive square roots above. (Note that the graphing software detects discontinuities - the straight horizontal lines. And so, the discontinuities also reflect that, for continuity, certain choices within the calculations would need to be made before hand. This particular software is not programmable in the sense that it can make decisions within a routine.)

For: $e^{2 \pi i \frac{1}{2}}$ meaning the negative square root relative to the three cubic roots; and, for the cube roots: $e^{2 \pi i \frac{1}{3}}$ in blue, $e^{2 \pi i \frac{2}{3}}$ in red, and $e^{2 \pi i \frac{3}{3}}$ in black.


Animation 21 'Cubic Polynomial Bifurcation 2'

If the polynomial has complex coefficients (see section below), turning points for any given polynomial that has them may lie in space. For example, with:

$$
x^{3}+(2+3 i) x^{2}-x i+(3-y)=0
$$

the local maximum can be seen here to be off the FRP and displaced in the positive iz direction.


When the cubic has only one real root for all portions of the graph, the other two roots will form branches wholly off the FRP:

$$
x^{3}+2 x^{2}+3 x+(3-y)=0
$$

In this example, the black line is the FRP graph:


### 4.62 Roots of a Quartic

Although more complicated, the same basic principles apply to the Quartic.

Graphing a more or less typical Quartic equation:

$$
x^{4}-2 x^{3}-2 x^{2}+2 x-2=y
$$

in two dimensions in the FRP:

we can see already where the bifurcation points will be: at the three turning points. These points will be where two or more of the four separate roots functions will meet.

Then, rearranging our equation:

$$
x^{4}-2 x^{3}-2 x^{2}+2 x-(2-y)=0
$$

using the quartic formula from (here), and using the same graphing procedure as for the cubic above and the quadratic previously, there are four separate roots functions generated. In the quartic formula there are square roots, cube roots and fourth roots, which, of course, makes a
too complicated picture to sort out here. So, the following four functions, taken from the procedure (here), would only be part of the total picture.

Note, as before, that separate roots functions contribute segments to the real FRP graph. (Also note where the graphing software detects discontinuities and tries to accommodate):


Animation 22 'Quartic Polynomial Bifurcation'

### 4.63 A Quintic Roots Graph

For the quintic below and the sextic following an online polynomial solver was used. Just as before if we take a quintic function:

$$
y=x^{5}-2 x^{4}-7 x^{3}+8 x^{2}+12 x
$$

the two dimensional FRP graph is:


The red horizontal lines enclose the interval:

$$
-12 \leq y \leq 12
$$

which is used for the 3Di roots graph below. $y$ takes on values on this interval in half unit steps and a numerical polynomial solver ${ }^{(*)}$ is used to find the five roots which are then graphed at the corresponding value of $y$.

So with the function converted to polynomial equation form:

$$
x^{5}-2 x^{4}-7 x^{3}+8 x^{2}+12 x-y=0
$$

and $y$ taking values on the interval in half steps, the quintic roots graph forms as follows:


## Rotate Quintic Roots Graph

Each of the dots is a single solution to the equation at its corresponding value of $y$. The black dots are the real solutions lying on the FRP and the blue dots are the complex solutions lying in space. Note the bifurcation points where the complex solutions and the real solutions meet. These meeting points or bifurcation points will move around relative to the coefficients in the original equation.

### 4.64 A Sextic Roots Graph

Continuing the same process for a sextic we take a typical sextic function:

$$
y=x^{6}+1.25 x^{5}+2.625 x^{4}+9 x^{3}-2 x^{2}-9.75 x
$$

and generate its two dimensional FRP graph:


As with the quintic above the red horizontal lines enclose the interval:

$$
-12 \leq y \leq 12
$$

which is used for the 3Di roots graph below. $y$ takes on values on this interval in half unit steps and the numerical polynomial solver ${ }^{(*)}$ is used to find the six roots which are then graphed at the corresponding value of $y$.

So converting the function to polynomial equation form:

$$
x^{6}+1.25 x^{5}+2.625 x^{4}+9 x^{3}-2 x^{2}-9.75 x-y=0
$$

and letting $y$ take values on the interval in half steps, using the numerical polynomial solver the sextic roots graph forms as follows:


The black dots, again, are the real solutions and the blue and red dots are the complex solutions. In this example the blue dots connect to the bifurcation points and the red dots form wholly independent branches.

### 4.65 Polynomial Nonlinearity

The definition that we have been using for nonlinearity is 'exponents other than 1 on the dependant variable'. We have been doing this in two ways:
A. In functions of the form $y^{n}=f(x)$ and,
B. In functions of the form $y=f(x) \rightarrow x+i z=g_{1,2,3}(y)$ where $g_{1,2,3}$ are the functions that result after applying the quadratic, cubic, and quartic equations which, in effect, reverse the roles of $x$ and $y$ making $y$ the independent variable and $x$ the dependant variable which had exponents other than 1 .

In the first form, (A) above, as with conics in section 2.61 Conic Nonlinearity, and Mordell and elliptic curves, section, we are just taking the $\mathrm{n}^{\text {th }}$ roots of $f(x)$ and applying Demoivre.
Numbers or the $\underline{n}^{\text {th }}$ roots of unity using:

$$
a+b i=e^{2 \pi i \frac{k}{n}}
$$

where $k$ is the $k^{\text {th }}$ root of the $n$ roots of unity. E.g., for the cube roots of unity, $n=3$ and $k=$ $1,2,3$. So, with a given function having an exponent other than 1 on the dependant variable:

$$
\begin{gathered}
y^{n}=f(x) \\
y=e^{\frac{2 \pi i k}{n}}(f(x))^{\frac{1}{n}}
\end{gathered}
$$

this makes $y$ complex so the individual roots functions result from each $k$ :

$$
y+i z=e^{\frac{2 \pi i k}{n}}(f(x))^{\frac{1}{n}}
$$

In the second form, (B) above, as said, we are reversing the roles of $x$ and $y$ by using the quadratic, cubic and quartic equations. This process generally makes $x$ complex and so we have:

$$
x+i z=e^{\frac{2 \pi i k}{n}}\left(g_{1,2,3 \ldots}(y)\right)^{\frac{1}{n}}
$$

And similarly, the individual roots result from each $k$. If techniques were available for solving quintic, sextic and so on equations, we could obtain the nonlinear roots graphs for $5^{\text {th }}, 6^{\text {th }}$ etc. degrees.

In equations of the first form there is theoretically no such limitation. For example, the sixth roots of a quintic:

$$
y^{6}=x^{5}-5 x^{3}+x^{2}+7 x-1
$$

which in two-dimensional form looks like this:

would, with all six roots:

$$
y+i z=e^{\frac{2 \pi i k}{6}}\left(x^{5}-5 x^{3}+x^{2}+7 x-1\right)^{\frac{1}{6}} \quad \text { for } k=1,2,3,4,5,6
$$

lying in space, look like the following. Each of the six roots is a different color.


## Animation 23 Polynomial Nonlinearity'

(Once again, the software has a little trouble with the bifurcation points.)

Also see section 7.1 Helix and Spiral Nonlinearity.

### 4.7 Complex Coefficients - Polynomials in Space

If we generate some simple polynomials with the binomial expansion, but include a constant imaginary term, as in:

$$
y+i z=(x+i)^{n}
$$

for $n=1,2,3,4$ :

$$
\begin{gathered}
y+i z=(x+i)^{1} \\
y+i z=(x+i)^{2}=x^{2}+2 i x-1 \\
y+i z=(x+i)^{3}=x^{3}+3 i x^{2}-3 x-i \\
y+i z=(x+i)^{4}=x^{4}+4 i x^{3}-6 x^{2}-4 i x+1
\end{gathered}
$$

the 3Di graphs are as follows. Notice that they lie in space with perhaps a single point on the FRP.

Degree $n=1$ :


Animation 24 'Polynomial in Space 1st degree'

## Degree $n=2$ :



Animation 25 'Polynomial in Space 2nd degree'

Degree $n=3$ :


Animation 26 'Polynomial in Space 3rd degree'

The RSIP view of this third degree is:

which is the historical curve known as The Tschirnhaus' Cubic, with equation:

$$
3 a y^{2}=x(x-a)^{2}
$$

With coefficient $a=5$, it is:


If $a$ is set to 9 and if the 3Di equation is translated with:

$$
y+i z=(x+i)^{3}+i
$$

then they become the same in two dimensions.

## Continuing to fourth degree with $n=4$ :

$$
y+i z=(x+i)^{4}=x^{4}+4 i x^{3}-6 x^{2}-4 i x+1
$$

Animation 27 'Polynomial in Space 4th degree'

### 4.8 Inverse Polynomial Functions

The inverses of these four examples are all asymptotic and resemble some of the historical curves in their side views:

For degree $n=-1$ :

This is the converging to zero circle along with the Witch and Serpentine that is in section 3.8 The Serpentine, Witch and a "Circle":


For degree $n=-2$ :
This is a converging to zero Cardioid in the side view:



For degree $n=-3$ :

This is a converging to zero Limacon in side view:



And for $n=-4$ :

This is a converging to zero Limacon with an additional crossing of the horizontal axis:



### 5.0 Complex Slope

3Di coordinates allow for a relatively simple extension of the standard concept of slope that we would now, in the front view, call 'real only' slope. To this is added 'imaginary only' slope which shows up in the top view. Summing the two together generates 'complex slope.'

### 5.1 An Intuitive Model



## Complex Slope

Using the animation above, visualize an aircraft taxiing down the runway prior to take-off. Our view is off to the side, with the taxiing aircraft moving from left to right. And, let's say that exactly to the right is a heading of zero. Exactly in front of us, the aircraft reaches take-off speed and rotates to begin its climb. This is the violet ball at the origin. The violet line is the aircraft's
climb while maintaining the same heading. This is real slope and zero imaginary slope, sometimes referred to as 'rise over run.'

Next, at the black ball, the aircraft reaches cruising altitude and levels off while maintaining the same heading. And, the black line shows its flight path with zero real slope and zero imaginary slope.

Next, at the red ball, the aircraft executes a 45-degree turn to the left while maintaining altitude. The red line shows its flight path with imaginary slope and zero real slope. This could be referred to as 'glide over run.'

And finally, while maintaining that heading, at the blue ball, it begins another climb. The blue line then shows both real slope, which is the climb, and imaginary slope, which is the heading other than zero. So, in flight path terms, complex slope is the sum of climb/descent plus heading.

### 5.2 Real, Imaginary and Complex Slope

Removing the idea of an aircraft, since it has a direction and motion, and just focusing on the line segments, real only slope of a line in three dimensions is:

while imaginary only slope is:


Real only slope appears in the front view (FRP) and imaginary only slope appears in the top view (TIP).

Real only slope has the usual slope-intercept equation of a line:

$$
y=m x+b_{y} \quad b_{y}=y \text { intercept }
$$

and imaginary only slope would then have a corresponding slope-intercept equation of a line:

$$
i z=i m x+i b_{i z} \quad i b_{i z}=i z \text { intercept }
$$

In graphing terms, real only slope is rise over run and imaginary slope would be glide over run. Either or both can be positive, negative or zero. Complex slope combines the two and is 'rise plus glide over run'. The two equations can be combined to give:

$$
y+i z=\left(m_{r}+i m_{i}\right) x+b_{y}+i b_{i z}
$$

Algebraically, complex slope extends standard slope by adding in the imaginary number for the glide. Since there are two slopes:
$m_{r}=$ the real component of complex slope, the rise in FRP im $_{i}=$ the imaginary component of complex slope, the glide in TIP

And the calculation of complex slope becomes:

$$
m_{r}+i m_{i}=\frac{\left(y_{2}+i z_{2}\right)-\left(y_{1}+i z_{1}\right)}{x_{2}-x_{1}}
$$

Also, rather than an axis intercept, there is a displacement of the line relative to both the $y$-axis and the $i z$-axis. Meaning there is a real displacement and an imaginary displacement. So the complete equation is:

$$
\begin{gathered}
y+i z=\left(m_{r}+i m_{i}\right) x+d_{r}+i d_{i} \\
d_{r}+i d_{i}=\text { complex displacement of the line }
\end{gathered}
$$

The real displacement moves the line up and down. The imaginary displacement moves the line forward and backward.

### 5.21 Example:

What is the equation of the line that goes through the two points: $(3,2, i)$ and $(1,-3,6 i)$ ?
The first step is to calculate the two slopes, real and imaginary:

$$
\begin{gathered}
m_{r}+i m_{i}=\frac{\left(y_{2}+i z_{2}\right)-\left(y_{1}+i z_{1}\right)}{x_{2}-x_{1}} \\
m_{r}+i m_{i}=\frac{(-3+6 i)-(2+i)}{1-3} \\
m_{r}+i m_{i}=\frac{-5+5 i}{-2} \\
m_{r}=\frac{5}{2} \quad i m_{i}=\frac{-5 i}{2}
\end{gathered}
$$

(The two slopes, of course, need not be equal. This example just turned out that way.)

The second step is to insert the slopes along with either point into the basic equation to solve for the displacements. Using the first point:

$$
\begin{gathered}
y+i z=\left(m_{r}+i m_{i}\right) x+d_{r}+i d_{i} \\
2+i=(2.5-2.5 i) 3+d_{r}+i d_{i} \\
2+i=(7.5-7.5 i)+d_{r}+i d_{i} \\
-5.5+8.5 i=d_{r}+i d_{i r}
\end{gathered}
$$

The third step, if needed, is to insert the slopes and the second point into the basic equation to verify that the two points give the same displacements.

$$
\begin{gathered}
y+i z=\left(m_{r}+i m_{i}\right) x+d_{r}+i d_{i} \\
-3+6 i=(2.5-2.5 i)+d_{r}+i d_{i} \\
-5.5+8.5 i=d_{r}+i d_{i}
\end{gathered}
$$

And so, the completed equation for the line with the two specified points is:

$$
\begin{gathered}
y+i z=\left(m_{r}+i m_{i}\right) x+d_{y}+i d_{i z} \\
y+i z=(2.5-2.5 i) x-5.5+8.5 i
\end{gathered}
$$

When this line is projected to the front view (real only slope) it appears as:


When the line is projected to the top view (imaginary only slope) it appears as:


The 3Di graph of the line along with the two specified points is as follows:

rotate complex slope line

### 5.3 Inverse Imaginary Slope

In addition to real slope in the FRP and imaginary slope in the TIP, denoted by:

$$
\begin{aligned}
& m_{r}+i m_{i}=\frac{\left(y_{2}+i z_{2}\right)-\left(y_{1}+i z_{1}\right)}{x_{2}-x_{1}} \\
& m_{r}=\frac{\left(y_{2}-y_{1}\right)}{x_{2}-x_{1}} \quad i m_{i}=\frac{\left(i z_{2}-i z_{1}\right)}{x_{2}-x_{1}}
\end{aligned}
$$

we can define an 'inverse imaginary slope':

$$
i_{i i}=\text { inverse imaginary slope }
$$

and denote it by:

$$
i m_{i i}=\frac{y_{2}-y_{1}}{i z_{2}-i z_{1}}
$$

Real slope shows up in the front view and imaginary slope shows up in the top view. Inverse imaginary slope shows up in the right side view.

One way that this can be visualized is by standing at the end of a runway while the aircraft takes off going away from us. In this front view the aircraft appears to rise vertically. This vertical ascent appearance occurs in both front and top views. And, this demonstrates that a line with inverse imaginary slope, so defined, appears as a vertical line in both the FRP and the TIP. Intuitively, we can carry these visualizations further to formally observe that:
(1) A line with real only slope shows up as a vertical line in the side view, and a line with zero slope in the top view.
(2) A line with imaginary only slope shows up as a line with zero slope in front view, and a line with zero slope in side view.
(3) And, as stated above, a line with inverse imaginary only slope shows up as a vertical line in both front and top views.

Inverse imaginary slope may appear somewhat counter intuitive in that the glide path of the above mentioned aircraft would have a positive inverse imaginary slope on landing/approach, and a negative inverse imaginary slope on take-off/departure.

Continuing the example with the two previously specified points, $(3,2, i)$ and $(1,-3,6 i)$, the inverse imaginary slope can be calculated as:

$$
\begin{gathered}
i m_{i i}=\frac{y_{2}-y_{1}}{i z_{2}-i z_{1}} \\
i m_{i i}=\frac{-3-2}{6 i-i}=\frac{-5}{5 i} \\
i m_{i i}=\frac{-1}{i} \quad \text { or } \quad i m_{i i}=i
\end{gathered}
$$

This can be viewed, in the above last animation, as the blue line comes around to show the RSIP view; and it can be verified by projecting the line (blue) to the RSIP as follows. In the side view the axes are:

$$
(\text { horizontal, vertical })=(i z, y)
$$

and the two displacements, when combined, project to a $y$-intercept that is different. By inserting the two points into the equation:

$$
y=i m_{i i}(i z)+b_{y}
$$

the $y$-intercept is calculated as:

$$
\begin{gathered}
\text { 1st point: }(i z, y)=(i, 2) \\
2=i(i)+b_{y} \quad 3=b_{y} \\
2 n d \text { point: }(i z, y)=(6 i,-3) \\
-3=i(6 i)+b_{y} \quad 3=b_{y}
\end{gathered}
$$

So the equation of this line is:

$$
y=i(i z)+3
$$

And when this line is projected to the right side view it appears as:


The three different two dimensional graphs are generated by the following equations:

> Front view $\quad y=m_{r} x+d_{r}$
> Top view $\quad i z=i m_{i} x+i d_{i}$
> Right side view $\quad y=i m_{i i} i z+b_{y}$

### 5.4 Table of Slopes in 3Di

Additionally, the three 2-dimensional slopes can also be visualized as the three planar rotations of a spacecraft. I.e., pitch, yaw and roll which is indicated in the fourth column of the table:

| slope | notation | plane | slope <br> rotation | action | 2D relationships |
| :--- | :---: | :--- | :--- | :--- | :--- |
| complex | $m_{r}+i m_{i}$ |  | pitch + <br> yaw | rise + glide <br> over run | Slope in all <br> three, FRP, TIP, <br> RSIP |
| real only | $m_{r}$ | FRP | pitch | rise over run | Horizontal in TIP, <br> vertical in RSIP |
| imaginary <br> only | $i m_{i}$ | TIP | yaw | glide over run | Horizontal in <br> both FRP and <br> RSIP |
| inverse <br> imaginary | $i m_{i i}$ | RSIP | roll | rise over glide | Vertical line in <br> both FRP and TIP |

### 5.5 Transformation of Two Dimensional Slope

## If we specify a point at $(1,1 / 2,0 i)$ :


and then draw a line through this point with no displacement, meaning a line through this point and the origin:


The equation of this line is generated by:

$$
y+i z=\left(m_{r}+i m_{i}\right) x+d_{r}+i d_{i}
$$

With no displacement, zero imaginary slope and an arbitrary $1 / 2$ real slope, the equation reduces to:

$$
y=\frac{x}{2}
$$

which is a line with real-only slope located on the front, real plane (FRP).

If we were to rotate this line about the $x$-axis using a 'rotator coefficient' $i^{a}$ (see sections: 3.25, 3.7, 9.38 on this rotator in the eBook), the effect is to alter the line's two dimensional real and imaginary slopes. Meaning:

$$
\begin{gathered}
m_{r}+i m_{i}=i^{a} \\
0 \leq a \leq 4 \\
y+i z=\frac{\left(m_{r}+i m_{i}\right) x}{2}=\frac{i^{a} x}{2}
\end{gathered}
$$

So as a moves through its interval, the line is rotated about the $x$-axis:


## line rotation about the $x$-axis

As the line rotates, its projected two dimensional slope transforms from real-only to complex to imaginary only to complex. And then to negative real-only to complex to negative imaginaryonly to complex and then back again to positive real-only. So if we look at various values of $a$ : For $a=0$ the slope is positive real only:

$$
y=\frac{x}{2}
$$

On the right, the black line is the two dimensional real slope in the front view and the red line is the two dimensional imaginary slope in the top view:



For $a=.6$ the slope is complex:

$$
y+i z=\frac{(.58779+.80902 i)}{2} x
$$



For $a=1$ the slope is positive imaginary only:

$$
i z=\frac{i}{2} x
$$



For $a=1.6$ the slope is complex with negative real and positive imaginary:

$$
y+i z=\frac{(-.80902+.29390 i)}{2} x
$$



For $a=2.6$ the slope is complex with negative real and negative imaginary:

$$
y+i z=\frac{(-.58779-.80902 i)}{2} x
$$



For $a=3$ the slope is negative imaginary only:

$$
i z=\frac{-i}{2} x
$$



As stated before, if a displacement is added to the line, a real displacement moves the line up and down and an imaginary displacement moves the line forward or backward. Then the line will rotate at the displacement point about a line (green line below) through that point and parallel to the x -axis. For example, adding a displacement and using the above line with $a=3$ :

$$
y+i z=\frac{-i}{2} x+1+i
$$



Just as this line, which is rotated about the x-axis, transitions between real and imaginary slope, if the line is rotated about the iz-axis the slope will transition between real and inverse imaginary slopes. Similarly if the line is rotated about the $y$-axis the line will transition between imaginary and inverse imaginary slopes.

### 5.6 Polynomial Space Trajectories

In section 4.7 Complex Coefficients - Polynomials in Space, we showed some examples of how an imaginary coefficient on the different terms will affect the graph of polynomials.

Taking a slightly different approach, if we, for example, simply add an $x^{2}$ term to the above equation of a line, giving:

$$
y+i z=x^{2}+(2.5-2.5 i) x-5.5+8.5 i
$$

the graph, in red, becomes a space parabola. The blue line and it's two points is the previous example.


[^3]As can be seen, adding the $x^{2}$ term bends the line into a parabola while keeping the same orientation in space. Further, the original line is tangent to it, which can be proven by setting the two equations equal, which gives zero as the value for $x$, and then the tangent point, in green, is just the displacement.


Seen in this way, complex coefficients on the terms of polynomials of virtually any degree and number of terms, are actually slopes that can generate three-dimensional trajectories along which objects can travel. Before we can do this, a few more variables are required.
(The colored balls in these examples are default objects of the PT-GC software, but in section 12.0 4Dii: Surfaces, we will generate our own spheres and other closed surface objects, like barrels and lozenges. Then, in section 15.0 6Dii: Surfaces in Motion, these objects will be put in motion along both orbits and polynomial space trajectories.)

# 6.0 Euler's Formula Upgraded, Helix and Spiral Functions 

### 6.1 Euler's Formula in Three Dimensions

In this section, we will go into more detail on bringing Euler's Formula into three dimensions and derive an upgraded version of it. Euler's formula is, of course, stated in base $e$ exponential form. His formula is now upgraded for all bases - positive, negative, real, imaginary and complex. Additionally, there are now four identities associated with the formula. The identities are also stated for all bases - positive, negative, real, imaginary and complex.

Normally, If we were to take Euler's identity, his 'Famous Five Equation,'

$$
e^{\pi i}+1=0
$$

which is a special case of:

$$
e^{i \theta}=\cos \theta+i \sin \theta
$$

if generalized to the usual complex plane:

$$
x+i y=e^{i \theta}
$$

and then, generalized further to 3Di coordinates:

$$
y+i z=e^{i x}
$$

and again, generalized further to include amplitude and frequency characteristics:

$$
\begin{gathered}
y+i z=A e^{a i x} \\
A=\text { ampltiude }, \quad a=\text { frequency }
\end{gathered}
$$

and then, generalized further still, to include multiple terms:

$$
y+i z=A e^{a i x}+B e^{b i x}+C e^{c i x}+\cdots
$$

This multiple term helix equation is what we were using in the previous sections: 2.7 Helixes in 3Di and 3.0 Historical Curves in 3Di. Any number of terms may be added such that we can generate what may be called a virtually limitless new class of 'Helix, Spiral and Rotating Exponential Functions'.

### 6.2 Coordinates in the Normal Complex Plane and in 3Di

The normal complex plane has coordinates ( $x$, iy). In 3Di coordinates, the Top Imaginary Plane (TIP) and the Right Side Imaginary Plane (RSIP) also have one axis imaginary and one axis real. They have coordinates, respectively, $(x, i z)$ and $(i z, y)$. Both of these are two-dimensional projections of any three-dimensional curve. Along with the Front Real Plane (FRP), these are the three main two-dimensional planar projections of curves in space. See section 2.3 Projection Planes.

Euler's formula generates a circle in the usual complex plane, but it generates a helix in 3Di as follows:

Beginning with Euler's wonderful Equation:

$$
e^{i \theta}=\cos \theta+i \sin \theta
$$

and then, by making it into a parametric function of $\theta$, we trace out a unit circle in the usual complex plane.

This unit circle then becomes periodic as $\theta$ increases past $2 \pi$, and decreases past $-2 \pi$, making it a multi-valued function in two dimensions.

The circle is formed parametrically in the complex plane by:

$$
\begin{aligned}
x & =\text { real } e^{i \theta} \\
i y & =\operatorname{imag} e^{i \theta}
\end{aligned}
$$

In 3Di, this unit circle shows up naturally in the RSIP (right side imaginary plane) as the circle which is a side view projection of a helix.

In three dimensions, as we generate the 'Euler Helix' and its three planar projections, the circle in the complex plane is the same as the circle in the RSIP, but with the axes switched. This is for the side view.

For the top view, the axes are the same for the complex plane and the TIP, meaning: horizontal real and vertical imaginary. But, since the TIP is a top view, meaning we are looking down on the curve, it would be as if we laid the complex plane down in front of us; and so, what was vertical in the complex plane becomes depth in 3Di. As we said in section 2.1 Constructing the Coordinate System, 3Di is constructed by taking both the real plane and the complex plane, joining them along the x-axis, and making them orthogonal to one another.

The conversion of the parametric 2Di coordinates in the normal complex plane, to the 'Euler Helix' in 3Di is:

3Di coordinate conversion

From: Parametric coordinates in the complex plane,

$$
\begin{gathered}
x+i y=e^{i \theta} \\
\begin{array}{c}
x=\operatorname{real} e^{i \theta} \\
i y=\operatorname{imag} e^{i \theta}
\end{array}
\end{gathered}
$$

To:
3Di

$$
\begin{aligned}
\theta & =x \\
x & =y \\
i y & =i z
\end{aligned}
$$

With the axes so redefined, we can now express Euler's Formula as a true three-dimensional function - two output numbers and one input number:

$$
y+i z=e^{i x}
$$

To review from section 2.2 A Left Hand System, the coordinates, $x, y$ and $i z$ are graphed as follows:
(1) The input $x$ will be graphed on the horizontal axis, left middle finger, positive is to the right.
(2) The real output $y$ will be graphed on the vertical axis, left thumb, positive is up.
(3) And the imaginary output $i z$ will be graphed on the 'depth' axis, left index finger, positive is forward, to the front.

Here, violet, is the FRP (front real plane) which, of course, is the normal cosine curve:


And, in red, the TIP (top imaginary plane), which of course, is the normal sine curve, but in 3Di its curve lies orthogonal to the FRP:


And, in blue, the RSIP (right side imaginary plane), which is the normal unit circle in the complex plane that in 3Di is both orthogonal to the other two graphs and has the axes switched. I.e., the imaginary axis is horizontal.


Here are the three planar projections in 3D:


And the 3Di helix, itself:


### 6.3 The Helix Base $\beta$ and Wavelength $\lambda$

The Euler Helix given by;

$$
y+z i=e^{i x}
$$

can be applied in a more general way to helixes of any base, $\beta$;

$$
y+i z=\beta^{i x}
$$

In a helix, altering the base changes the wavelength. But, most often, the e-base is retained, and wavelength and frequency characteristics are changed by adding a coefficient to the exponent. I.e.

$$
y+i z=e^{f i x} \quad f=\text { frequency }, \quad \frac{2 \pi}{f}=\text { wavelength }=\lambda
$$

This is, of course, fine for many applications. But, keeping the concept of different bases allows for an easier understanding of the greatly expanded range of helix and spiral functions that come with negative, imaginary and complex bases, as well as negative, imaginary and complex
wavelengths - all of which, geometric interpretations will follow. Rather than only frequency changes to the e-base helix, the relationship between base and wavelength will serve to easier analyze what's going on with all of the new and different possibilities.

To derive this broader view of the base/wavelength relationship, we go back to Euler's identity:

$$
e^{\pi i}+1=0
$$

and, rearrange:

$$
-1=e^{\pi i}
$$

and, take natural logs without simplifying:

$$
\ln (-1)=\pi i(\ln e)
$$

Then we can observe that, in this particular case, $e$ is the base $\beta$, and $\pi$ is the wavelength divided by 2. I.e., $\frac{\lambda}{2}$. So, a more general statement of Euler's identity can be written:

$$
\ln (-1)=\frac{\lambda i}{2}(\ln \beta)
$$

Since the natural $\log$ of $e$ is equal to 1 , It is still true that;

$$
\ln (-1)=\pi i
$$

And so:

$$
\pi i=\frac{\lambda i}{2}(\ln \beta)
$$

Simplifying and rearranging we have the direct base, wavelength relationship:

The Base - Wavelength Relationship

$$
\lambda \ln \beta=2 \pi
$$

Or:

$$
\lambda=\frac{2 \pi}{\ln \beta}
$$

which effectively expands the definition of wavelength to include negative, imaginary and complex bases in addition to real ones.

Note that in the form:

$$
\ln \beta=\frac{2 \pi}{\lambda}
$$

this is the same as the wavenumber equation ${ }^{*}$ *

$$
k=\frac{2 \pi}{\lambda}
$$

And so, in this context, wavenumber is, in fact, equal to the natural log of the base!

$$
k \equiv \ln \beta
$$

And, when needed, we can write this in exponential form:

$$
\beta=e^{\frac{2 \pi}{\lambda}}
$$

### 6.31 The New Famous Five Identities

Noting in Euler's identity:

$$
e^{\pi i}+1=0
$$

as we did above, that in this particular case, $e$ is the base $\beta$, and $\pi$ is the wavelength divided by 2 , i.e. $\frac{\lambda}{2}$, so a more general statement of Euler's identity, can be written by taking from above:

$$
\ln (-1)=\frac{\lambda i}{2}(\ln \beta)
$$

and then converting this back to exponential form:

$$
-1=\beta^{\frac{\lambda i}{2}} \rightarrow \beta^{\frac{\lambda i}{2}}+1=0
$$

then, we have stated Euler's identity for any base and its associated wavelength, this tells us that Euler's Famous Five equation is true for all bases! This means that when any helix base $\beta$ has its wavelength $\lambda$ in the exponent, the result is always equal to 1 ; and, if the exponent is equal to the wavelength over 2 , i.e., $\lambda / 2$, the result is always equal to -1 .

Additionally, there is an identity for $\lambda / 4$ and $3 \lambda / 4$ so:

The New 'Famous Five' Identities:

$$
\begin{gathered}
\beta^{i \lambda}-1=0 \\
\beta^{i \lambda / 2}+1=0 \\
\beta^{i \lambda / 4}+i=0 \\
\beta^{3 i \lambda / 4}-i=0
\end{gathered}
$$

This will be seen to be true for all base-wavelength pairs including real, imaginary and complex.

For example, the $i$-base:

$$
\lambda=\frac{2 \pi}{\ln \beta}
$$

$$
\begin{gathered}
-4 i=\frac{2 \pi}{\ln i} \\
i^{i(-4 i) / 2}+1=0
\end{gathered}
$$

And, a random complex base $(\pi+i)$ :

$$
\begin{gathered}
\lambda=\frac{2 \pi}{\ln \beta} \\
4.93732-1.2754 i=\frac{2 \pi}{\ln (\pi+i)} \\
(\pi+i)^{i(4.93732-1.2754 i) / 2}+1=0
\end{gathered}
$$

### 6.32 Euler's Famous Formula Upgraded

Euler's formula:

$$
e^{i \theta}=\cos \theta+i \sin \theta
$$

combined with the laws of exponents and de Moivere's theorem ${ }^{* *}$ :

$$
\begin{gathered}
\left(e^{i \theta}\right)^{n}=\cos n \theta+i \sin n \theta \\
e^{n i \theta}=\cos n \theta+i \sin n \theta
\end{gathered}
$$

allows for some interesting insights. First, if $n=1$ we have, of course, the 'Euler Helix' that we have referenced several times.

In FRP:


Next, if $n=\lambda=2 \pi$ in the FRP:

we have a helix with both wavelength and frequency equal to 1 . This will be seen to be true any time a base and wavelength pair are used for a helix, meaning:

$$
y+i z=\beta^{\lambda i x}
$$

gives a helix with wavelength and frequency $=1$. So, for example, an arbitrary helix, referencing the above complex base:

$$
y+i z=(\pi+i)^{(4.93732-1.2754 i) i x}
$$

will also give a helix with wavelength and frequency equal to 1 .

Now, the fun part: Just as when we derived the initial base-wavelength relationship, we observed a deeper meaning in $\ln e=1$, and then made use of that fact. So, in the equation above which used the laws of exponents and de Moivere's theorem:

$$
e^{n i \theta}=\cos n \theta+i \sin n \theta
$$

When $n=1$, it is also true that $n=1=\ln e$. We then have:

$$
e^{(1) i \theta}=\cos (\theta \ln e)+i \sin (\theta \ln e)
$$

And then, just as with the base-wavelength relationship, we can speculate that this is true for any base. If so, we have:

## The Euler formula upgraded for any base $\beta$ :

$$
\beta^{i \theta}=\cos (\theta \ln \beta)+i \sin (\theta \ln \beta)
$$

Or, if the helix is regarded as a wave, since the wavenumber $k=\ln \beta$

$$
\beta^{i \theta}=\cos (k \theta)+i \sin (k \theta)
$$

And then, in wavelength form:

$$
\begin{gathered}
\beta^{i \theta}=\cos \left(\frac{2 \pi \theta}{\lambda}\right)+i \sin \left(\frac{2 \pi \theta}{\lambda}\right) \\
\beta=a+b i
\end{gathered}
$$

a or $b$ may be zero but not both.

So, except for $\beta=0+0 i$, 'Euler's Formula Upgraded' makes his formula valid for all bases positive, negative, real, imaginary or complex.

A relatively simple proof for:

$$
\beta^{i \theta}=\cos (\theta \ln \beta)+i \sin (\theta \ln \beta)
$$

begins by substituting:

$$
p=\theta \ln \beta
$$

on the right side giving:

$$
\beta^{i \theta}=\cos p+i \sin p
$$

and then converting the left side to $e$ base by solving for $y$ in:

$$
\beta^{i \theta}=e^{y}
$$

giving:

$$
y=i \theta \ln \beta
$$

so:

$$
y=i p
$$

which gives:

$$
e^{i p}=\cos p+i \sin p
$$

Euler's formula itself which, of course, is already proved.

### 6.4 Wavelength Interpretations

The wide ranging number of possibilities, which includes: amplitude characteristics, geometric possibilities, additional terms, and additional levels of exponents, will result in graphs of general types:
(A) Exponential Graphs
(B) Rotating Exponential Graphs
(C) Helixes
(D) Spirals
(E) Combinations of the above

Additionally, each of these will have two or more general characteristics from among:
(1) Axis location: Existing on the real axis, imaginary axis, somewhere in between, or in space.
(2) Amplitude: Continuously increasing, asymptotic, or constant.
(3) Frequency/Wavelength: $\lambda$ may be positive, negative, real, imaginary, or complex (Definitions below.)
(4) Direction: Positive or negative, real, imaginary, or complex.
(5) Rotational Development: Clockwise (CW) or counterclockwise (CCW.)

The following is a very incomplete list of a few possibilities. And, as can be seen, the table itself is incomplete. Its purpose is only to serve as a general organizational beginning.

The first column lists several base possibilities. The second column describes the effect of a variable exponent: i.e., $x$, if it is real. The third column describes the effect of that variable exponent if made imaginary. There would be, in addition, exponents that may have part variable and part constant, both constant (for variable base), both variable, and so on.

And, the fourth column is the type of wavelength that results using that particular combination of base and wavelength relationship.
6.41 Table of Exponent Effects for Various Bases:

| Constant Base $\beta$ | Variable Exponent Real Part | Variable Exponent Imaginary Part | Wavelength $\lambda$ |
| :---: | :---: | :---: | :---: |
| Real Positive | Exponential: increasing values for positive $x$ | Helix: CCW | positive real |
| Real Positive Inverse | Exponential: increasing values for negative x | Helix: CW | negative real |
| Real Negative other than (-1) | Spiral: expanding outward from helix at $\beta=-1$ CCW | Spiral contracting inward from an exponential at $\beta=-1$ and increasing values for negative $x$ | complex (+Re, -Im) |
| (-1) | Helix: CW | Exponential: increasing values for negative x | -2i |
| $i$ | Helix: CW | Exponential: increasing values for negative $x$ | $-4 i$ |
| $-i$ | Helix: CCW | Exponential: increasing values for positive $x$ | $4 i$ |
| Imaginary Positive other than $i$ | Spiral with increasing amplitude for positive $x$ | Spiral with increasing amplitude for negative x | complex (+Re, -Im) |
| A few Additional <br> Base possibilities <br> below |  |  |  |
| Imaginary <br> Negative |  |  |  |
| Real Negative Inverse |  |  |  |
| Variable Real Base $\beta=x$ |  |  |  |
| Variable Complex <br> Base $\beta=(x+b i)$ <br> or $\beta=(a+w i)$ <br> or $\beta=(x+w i)$ |  |  |  |

### 6.42 Positive Real Wavelengths

This is the normal or usual situation which has been used thus far, except for a few exceptions like the rotating exponential and spiral ( section 2.8 Exponentials in 3Di), and some of the historical curves (section 3.6 Spira Mirabilis 6 New 3Di Equations for the Equiangular Spiral Here are a few more multiple term examples:

Loops and More Loops Helix:

$$
\begin{gathered}
y+i z=A e^{a i x}+B e^{b i x}+C e^{c i x}+D e^{d i x} \\
a=1, b=6, c=-1, d=2, A=0.8, B=1.05, C=1.825, D=-1
\end{gathered}
$$


greg ehmka, 2013
"Stop Sign" Helix:

$$
\begin{gathered}
y+i z=A e^{a i x}+B e^{b i x} \\
a=-1, b=7, A=2.6, B=-0.1
\end{gathered}
$$



"Triangle" Helix:

$$
\begin{gathered}
y+i z=A e^{a i x}+B e^{b i x} \\
a=-1, b=2, A=1.9, B=0.5
\end{gathered}
$$

## "Triangle" Helix




### 6.43 Negative Wavelength Interpretation

If we happen to have helix base values of $0<\beta<1$, The natural log of that value will result in wavelengths that will be negative. We can interpret the difference between a negative and a positive wavelength in the following way: as a CCW (counter clockwise) development of the helix for positive wavelength, and as a CW (clockwise) development for negative wavelength values.

This would be as follows: standing at the origin with "forward" directed along the positive $i$, or imaginary (depth) axis, and then, making a quarter turn to the right. We would now be facing toward positive $\infty$ along the real ' $x$ ' axis. This is equivalent to the left side view (LSIP.) With this orientation, positive wavelength develops CCW and negative wavelength develops CW.

For example, the two graphs below have reciprocal bases and equal wavelengths, except for the sign. We can see that the positive wavelength (left graph) has an initial development CCW into the octant, which is above the origin plane and to the left; whereas, the negative wavelength (right graph) has an initial development into the octant above the origin plane and to the right.

Visually, imagining ourselves standing with this orientation, i.e., standing at the origin facing positive horizontal infinity, the positive wavelengths develops CCW into the forward upper left octant, whereas negative wavelengths develop CW into the forward upper right octant.
$y+i z=(3)^{i x} \quad \lambda=5.719201 \ldots \quad y+i z=(1 / 3)^{i x} \quad \lambda=-5.719201 \ldots$

Interval: $-1 \leq x \leq 1$


Interval: $-2 \leq x \leq 2$


Due to the symmetry of the helix, it is important to note that this difference cannot be seen in the normal real plane, the FRP. It also cannot be seen in either of the RSIP or LSIP side views if the interval for ' $x$ ' extends in both the positive and negative direction. It is easiest seen in the TIP:


This difference is able to be seen in the side view if we make the interval for $x$ small and positive.

In LSIP, $0 \leq x \leq 2$


6.44 Imaginary Wavelength Interpretation

Using the base-wavelength relationship:

$$
\lambda=\frac{2 \pi}{\ln \beta}
$$

wavelengths that are imaginary only, can be generated with bases:

$$
\beta=i,-1,-i
$$

Then:

$$
-4 i=\frac{2 \pi}{\ln (i)}, \quad-2 i=\frac{2 \pi}{\ln (-1)}, \quad 2 i=\frac{2 \pi}{\ln (-i)}
$$

Comparing with the 'Euler Helix,' $y+i z=e^{i x}$ which has real only wavelength of $2 \pi$, a real wavelength (in black) lies along the real x -axis:

while an imaginary only wavelength (in red) with:

$$
\beta=-1 \text { and } y+i z=(-1)^{x}
$$

can be defined to lie along the imaginary or depth axis. In this case, with a wavelength of 2 :

As before, the sign indicates CW for negative and CCW for positive. Except, in this case, we are standing at the origin and facing positive, imaginary infinity, $+i \infty$. With this orientation the development is CW.

Note that the exponent is imaginary with $\beta=e$, and real with $\beta=-1$. Further, note that by defining an imaginary wavelength to lie along the depth axis, we are defining the input to be graphed along the $i z$, or depth axis, and then the two outputs are graphed on the $y$ and $x$ axes respectively.

Note further that this means we are graphing an imaginary value along the x -axis! So, we have defined an imaginary wavelength to, in effect, occur within a different coordinate system. That being 3Dii, rather than 3Di. They are actually different versions of 4Dii coordinates.

For the helixes, in both the $e$-base and the $i$-base cases, the input values are actually the angle values $\theta$. So, the functions are more accurately described by:

$$
\begin{gathered}
y+i z=e^{i \theta} \\
y+i x=i^{\theta}
\end{gathered}
$$

And then, $\theta$, as the input, is defined to be either on the $x$-axis or $i z$-axis, in accordance with the wavelength being either real or imaginary.

Along with a new geometry of natural logarithms, section 11.22 The Four Coordinate Complex Exp/Log Function, this defined 3Dii coordinate system will be seen in section 11.3 The

Imaginary Logarithmic Surface, to set the stage for a geometric interpretation of imaginary logarithms!

### 6.45 Complex Wavelength Interpretation

A complex value for the wavelength $\lambda$ essentially indicates a spiral. The quantity lambda $\lambda$ has some interesting characteristics. For positive real bases it does indeed indicate the actual wavelength, which we can call, in two dimensions, the 'pitch' of the helix or spiral.

For bases, $\beta=i,-1,-i$, it also indicates the pitch, except that it adds the two additional pieces of information: that the helix or spiral runs along the imaginary axis, and whether it develops CW or CCW.

For all other bases, where $\lambda$ becomes complex and a spiral results, it may indicate both pitch and amplitude or only amplitude!

Helixes become spirals with gradual changes to the base. In the following animation the base $\beta$ is changing for:

$$
y+i z=\beta^{x}
$$

The interval is: $-.5 \leq \beta \leq-1.5$ Although the base values do not show in the video, note that in the middle, when $\beta=-1$, a helix results; and, when the base is any other value a spiral results.


[^4]Next, notice that in the following two-dimensional animation of the above, the pitch does not change. The base-wavelength relationship gives a complex value for $\lambda$ that only applies to the amplitude! Note that the pitch remains fixed and equal to 2 .
The real part in violet and the imaginary part in red:


Animation 30 'Negative Base Spiral Morphing with Constant Pitch'

In the next example, the base $\beta$ is negative real $=-1$, and becomes complex by virtue of the presence or absence of a constant imaginary addition.

$$
\begin{gathered}
y+i z=(-1+n i)^{x} \\
-1.5 \leq n \leq 1.5
\end{gathered}
$$

When $n=0, \beta=-1$, as before, a helix results. For all other values of $n, \beta$ is complex and a spiral results. Note that, in this case, the spiral does not change direction at $n=0$, but rather changes development, moving from CCW for $n$ positive to CW for $n$ negative:


Animation 31 'Complex Base Spiral Morphing'

And, now, in the two-dimensional animation of the above, note that the pitch is changing. And, here, the complex value for $\lambda$ applies to both pitch and amplitude:


Animation 32 'Complex Base Spiral Morphing with Variable Pitch'

So, in terms of nomenclature, the quantity $\lambda$ which we have been calling 'wavelength' refers at times strictly to pitch, strictly to amplitude and sometimes both. Likely there are additional properties to discover. At least for spirals, $\lambda$ shows up, for the most part, as a constant ratio of the coordinates $(x, y, i z)$ as follows:

$$
y+i z=\beta^{x} \text { for spirals }
$$

$$
\begin{gathered}
y+i z=e^{\frac{2 \pi x}{\lambda}} \\
\ln (y+i z)=2 \pi x / \lambda \\
\lambda=\frac{2 \pi x}{\ln (y+i z)}
\end{gathered}
$$

"For the most part" means there are additional subtleties to be worked out in detail that have to do with:

- when the real and imaginary parts change places,
- the change in CW or CCW development,
- and with which bases either the real or the imaginary part of the exponent generates the spiral


### 6.5 Exponentials

### 6.51 The i-base Exponential

As is commonly known, $i^{i}$ is a real number and equal to .20787958 which is the reciprocal of $e^{\frac{\pi}{2}}$. As it turns out, any value for $z$ in:

$$
y=i^{i z}
$$

is also a real number. Graphing all of the values for $z$ results in the i-base exponential graph:


Contrasted with the usual e-base $y=e^{x}$ exponential graph, in black:

considering that the $y$-axis, vertical, is real in both cases; but that the input $x$, horizontal, is real for the e-base exponential; and the input $i z$ is imaginary for the i-base exponential; it's possible that the i-base exponential can be defined to be in the depth direction. If so, then in three dimensions:


They both may be made to have different 'slopes' with the coefficient $a$ :

$$
\begin{aligned}
& y=i^{a i z} \\
& y=e^{a x}
\end{aligned}
$$

With $a=.2$ :


### 6.52 Rotating Exponentials

The usual e-base exponential graph:

$$
y=e^{x}
$$

when given a complex exponent:

$$
y+i z=e^{x+i \theta}
$$

can be given a new interpretation. The new interpretation is that of a rotating exponential graph also shown in sections 1.3 Example, and in section 2.81 Exponential Graph Rotation. In the case of the e-base exponential, $\theta$ is imaginary, which would rotate the graph in the imaginary, or depth, direction as $i \theta$ takes on different values. This is an example of a function in 4Dii coordinates.

Interval: $0 \leq \theta \leq 4 \pi$ and back again:


Animation 33 'e-base Exponential Rotation'

Contrasted with the $i$-base exponential:

$$
y=i^{i z}
$$

if given a complex exponent:

$$
y+i z=i^{\theta+i z}
$$

$\theta$ is real. This suggests that the rotation for the $i$-base exponential is in the FRP. It is also a 4Dii function generating complex imaginary logarithms. See section 11.3 The Imaginary Logarithmic Surface.

The $i$-base exponential completes one rotation in units of $4-$ rather than $2 \pi$-and so with interval $0 \leq \theta \leq 8$, and back again:


Animation 34 'i-base Exponential Rotation'

The rotating exponential sets the stage for a new geometry of logarithms, both natural and imaginary. See sections 11.2 A New Geometry of Natural logarithms, and 11.3 The Imaginary Logarithmic Surface.

### 6.53 Exponential Tangents

There are some interesting ways that a real exponential graph is tangent to a spiral.

The first is the exponential of any positive real base, and the spiral resulting from the negative of that base.

$$
\begin{gathered}
y=2^{x} \\
y+i z=(-2)^{x}
\end{gathered}
$$

The exponential graph (in blue) is tangent to the real values of the spiral (in violet) on the even integers, and to the imaginary values of the spiral (in red) on the positive half-integers.


The negative of the exponential graph,

$$
y=-2^{x}
$$

is tangent to the real values of the spiral on the odd integers, and tangent to the imaginary values on the odd half-integers.


And, in three dimensions:


### 6.54 The Elliptic Spiral

Recalling from section 3.2 The 'Cusped Helix', a helix can be made to have any number of 'cusps', by different 'frequency coefficients', in the exponents of the reciprocal sum. And, with the elliptic helix, the amplitudes must be different. The same is true for spirals. For example:

$$
y+i z=(-1.05)^{x}+2(-1.05)^{-x}
$$

gives an elliptic spiral, in black, that gradually becomes circular as $\pm x$ increase. The two exponential tangent lines are in blue:

$$
\begin{gathered}
y+i z=(1.05)^{x}+2(1.05)^{-x} \\
y+i z=-\left((1.05)^{x}+2(1.05)^{-x}\right)
\end{gathered}
$$

Interval: $-20 \leq x \leq 24$


Animation 35 'Elliptic Spiral'

In two dimensions, the two exponential tangent lines only touch the real values (in violet) positive for the even integers, and negative for the odd integers. What is interesting is that it appears only the imaginary values (in red) are changing to alter the shape from elliptic to circular.


The end view of the negative $x$ half portion of the graph shows the change in shape more clearly. Interval: $-30 \leq x \leq 0$


A 'Triangular Spiral' or a 'Square Spiral' results from a change in the first exponent:

$$
\begin{array}{cc}
y+i z=(-1.05)^{2 x}+(-1.05)^{-x} & \text { Triangular Spiral } \\
y+i z=(-1.05)^{3 x}+(-1.05)^{-x} & \text { Square Spiral }
\end{array}
$$

The end views with the same interval:



As in section 3.2 The 'Cusped Helix', for helixes with different amplitude coefficients, the 'triangle' and the 'square' can be shaped with cusps or loops. In fact, the spiral shows cusps for negative $x$ and loops for positive $x$.

For the Triangular Spiral, only the positive exponential graph appears to be tangent, and only for the real values at the even integers:


For the Square Spiral, the negative exponential is again tangent: positive for even integers and negative for odd integers. And, both exponentials are again tangent to the imaginary values, but only certain ones:


The three-dimensional views:


### 6.6 The $x$ or Variable Base

A variable base or ' $x$ base' is, of course, that which generates polynomials. For some interesting variations see:

Section 4.7 Complex Coefficients - Polynomials in Space.
Section 7.31 2nd Level Exponents and the 'x-base'
Section 11.4 More Exponential/Logarithmic Surfaces, for $x$-base surface.

In this section there are a few examples of the helixes and spirals generated using the 'x base' with real and imaginary exponents. If the exponent is also real and variable, meaning:

$$
y+i z=x^{x}
$$

then, although undefined at $x=0$ there is an exponential graph for positive $x$, and a spiraling asymptotic curve graph for negative $x$ :


And, in the two-dimensional projections: real values, FRP (in violet), imaginary values, TIP (in red) and the side RSIP (in blue):


If the exponent is imaginary:

$$
y+i z=x^{x i}
$$

then there is a helix with decreasing wavelength for positive $x$, and an expanding spiral for negative $x$;


The two dimensional projections:


With an inverse real exponent:

$$
y+i z=x^{-x}
$$

The curve is not continuous at zero, is an expanding spiral for negative $x$ and is linearly asymptotic, i.e., not spiraling, for positive $x$ :



And, with an inverse imaginary exponent:

$$
y+i z=x^{-i x}
$$

the curve is not continuous at zero, is a helix having decreasing wavelength for positive $x$, and is spiraling asymptotic for negative $x$ :


## $7.02^{\text {nd }}$ and Higher Level Exponents

### 7.1 Helix and Spiral Nonlinearity

As with conic and polynomial nonlinearity (section 4.65 Polynomial Nonlinearity), 'Demoivre Numbers', or the $\mathrm{n}^{\text {th }}$ roots of unity, are used for the roots graphs of helixes and spirals: So, for the cube roots of the Euler Helix:

$$
y^{3}=e^{i x}
$$

the three roots graphs will be:

$$
y+i z=e^{2 \pi i \frac{k}{3}}\left(e^{i x}\right)^{\frac{1}{3}} \text { for } k=1,2,3
$$

Separately, the graphs have jump discontinuities. (And, as before the straight lines are the graphing software's attempt to connect the graph across the discontinuities, and can be ignored.) First root in violet, second root in red, third root in blue:



All together the three separate roots connect exactly at the jump discontinuity points:


As with the branches of the polynomial roots graphs, there are three cube root helixes generated that are each continuous by virtue of piecing together contributions from the actual roots graphs. These 'virtual helixes' go from violet, to red, to blue, and repeat.

The wave length of the virtual helixes is tripled to $6 \pi$. In two dimensions, original helix, in black:


If each of the roots graphs is cubed, or the three are multiplied together, the original helix is restored:


Possibly the virtual helixes and 'virtual spirals', next, will provide further insight to jump discontinuities in general.

A negative base spiral follows the same pattern:

$$
\begin{gathered}
y^{3}=(-2)^{t} \\
y+i z=e^{2 \pi i \frac{k}{3}}(-2)^{\frac{1}{3}} \text { for } k=1,2,3
\end{gathered}
$$

The original spiral:


The three separate spiral roots graphs, violet, red and blue forming 'virtual spirals' that move from violet to red to blue and repeating:


The original spiral has a two-dimensional pitch of 2, as do all negative base spirals:


And, the virtual spirals have that pitch tripled to 6 while the separate segments have a 'segmented wavelength' of 2.


The next two examples - one with square roots, and one with cube roots - are of an Airy Type spiral. See section 7.5 'Airy Type' Spiral for detailed discussion of Airy Type helixes. By Airy Type, what is meant, is an exponential graph for positive $x$, and periodic with decreasing amplitude for negative $x$. An Airy Type spiral can be generated with a second level exponent on the Euler Helix. I.e.,

$$
y+i z=e^{x^{c}} \quad c=1.475
$$

which, on the FRP projection, with interval: $-12 \leq x \leq 4$ has real values:


And, in three dimensions:


Animation 36 'Airy Type Helix'

And then, here is the same equation in $2^{\text {nd }}$ degree nonlinear form, taking square roots. For square roots, the roots of unity are the real numbers $\pm 1$ :

$$
(y+i z)^{2}=e^{x^{1.475}} \rightarrow y+i z= \pm\left(e^{x^{1.475}}\right)^{\frac{1}{2}}
$$

The original helix (in gray), plus both square roots together (in aqua):


Animation 37 ' $\mathbf{2 n d}$ Degree Nonlinear Airy Type Helix Roots'
Making the original equation into a $3^{\text {rd }}$ degree nonlinear form, and taking cube roots:

$$
(y+i z)^{3}=e^{x^{1.475}} \rightarrow y+i z=e^{2 \pi i \frac{k}{3}}\left(e^{x^{1.475}}\right)^{\frac{1}{3}} \text { for } k=1,2,3
$$


greg ehmka, 2013

The three cube roots together (in aqua), and the original helix (in gray):


Animation 38 ' 3 rd Degree Nonlinear Airy Type Helix Roots'

### 7.2 Additional Levels of Exponents

With higher levels of exponents an entire universe of exponential functions becomes extraordinarily interesting and limitless. 3Di coordinates allow for virtually any equation, no matter how unusual to "make sense' in some way.

The interplay between exponential graphs, helix graphs and spiral graphs all with multifaceted changes to amplitude, frequency, wavelength and so on make for an orderly and extraordinary array of possibilities.

Here is an arbitrary, somewhat radical, example of added levels of exponents. Note the tracing coordinates show that the graph is asymptotic to the base, $\pi$, in the positive $x$ direction:



Slightly altering the above equation so that the graph is asymptotic to $\pi$ in both directions:

$$
y+i z=\left(\pi^{\left(c+\left(\text { reale } e^{(i x)^{s}}\right)\left(x^{p}\right)^{s}\right)}\right) \quad c=1, s=1.11, p=1.25
$$

With FRP, real values, in violet and TIP, imaginary values in red:


These two graphs are orthognal to one another so in three dimensional, 3Di coordinates:


Zooming out and slightly rotating for the 3D helix;


And the, side view, RSIP projection to complete the picture;

In addition to being interesting in and of itself, this side view, planar projection demonstrates in an astonishing way just how much information can be lost in projecting a three dimensional object onto two dimensions and additionally how confusing it can be.

### 7.3 Variable Wavelength and Location of the 'Zeros'

In certain helixes and spirals with a normally fixed pitch there is a wonderful relationship between a second level exponent and the pitch/wavelength. When the second level exponent is 1 the usual fixed pitch results. And when the second level exponent is other than 1 the pitch will vary but in a regular way such that the location of the zeros can be predicted. There are many aspects to the process depending on the base, whether the zeros are the real ones or the imaginary ones and differences for positive $x$ and negative $z$. A few will be explored.

Using the $i$-base helix at first since its wavelength of 4 is easier to work with.

The usual $i$-base helix with its pitch of 4 :

$$
y+i z=i^{x}
$$



Adding a second level exponent less than one will make the wavelength longer and variable:

$$
y+i z=i^{x^{\frac{1}{\bar{a}}}}
$$

So with $a=2$ :

$$
y+i z=i^{x^{\frac{1}{2}}}
$$

For negative $x$ the graph is non spiraling asymptotic so there are no wavelengths:


For positive $x$ the beginning zero is at:

$$
\left(x_{0}, 0_{0}\right)=(1,0)
$$

And subsequent real zeros are at:

$$
\operatorname{real}\left(x_{n}, 0_{n}\right)=\left(x_{0}+2 n\right)^{2}
$$

Calculating the first few:

$$
\begin{aligned}
& \text { real }\left(x_{1}, 0_{1}\right)=(9,0) \\
& \text { real }\left(x_{2}, 0_{2}\right)=(25,0) \\
& \text { real }\left(x_{3}, 0_{3}\right)=(49,0) \\
& \text { real }\left(x_{4}, 0_{4}\right)=(81,0)
\end{aligned}
$$

And so on. For $a=3$ :

$$
y+i z=i^{x^{\frac{1}{3}}}
$$

The real zeros are at:

$$
\begin{aligned}
\left(x_{0}, 0_{0}\right) & =(1,0) \\
\operatorname{real}\left(x_{n}, 0_{n}\right) & =\left(x_{0}+2 n\right)^{3}
\end{aligned}
$$

The zeros and consequently the wavelength of any subsequent wave segment is a direct function of the second level exponent. And so:

$$
\begin{gathered}
y+i z=i^{x^{\frac{1}{a}}} \\
\operatorname{real}\left(x_{n}, 0_{n}\right)=\left(x_{0}+2 n\right)^{a}
\end{gathered}
$$

Since this particular curve is non spiraling, asymptotic for negative $x$ there are also no imaginary zeros for negative $x$. So for the imaginary zeros and positive $x$, the process is the same except there is no 'beginning zero.'

$$
\begin{aligned}
& \text { real }\left(x_{n}, 0_{n}\right)=\left(x_{0}+2 n\right)^{-a} \quad \text { in violet } \\
& \quad \text { imag }\left(x_{n}, 0_{n}\right)=(2 n)^{-a} \quad \text { in red }
\end{aligned}
$$



The same process works for integer exponents and rational exponents greater than 1 , so the exponents for the helix and for the zeros are simple inverses of each other:

$$
\begin{gathered}
y+i z=i^{x^{a}} \\
\left(x_{n}, 0_{n}\right)=\left(x_{0}+2 n\right)^{a^{-1}}
\end{gathered}
$$

Helix exponents greater than 1 shorten the pitch, and exponents less than 1 lengthen the pitch. So, for example, with $a=2$ :

$$
y+i z=i^{x^{2}}
$$

there are helixes in both directions of $x$ that mirror each other and have decreasing pitch/increasing frequency:


The zeros will be at:

$$
\begin{aligned}
& \operatorname{real}\left(x_{n}, 0_{n}\right)=\left(x_{0}+2 n\right)^{-2} \\
& \quad \operatorname{imag}\left(x_{n}, 0_{n}\right)=(2 n)^{-2}
\end{aligned}
$$



For positive real bases the helixes are similar to the imaginary base ones, i.e., helix in both directions for $a>1$, and non spiraling asymptotic for negative $x$ with $a<1$. The process for the zeros is slightly different.

For example, with e-base and $a=2$ :

$$
\begin{gathered}
y+i z=e^{i x^{2}} \\
\left(x_{n}, 0_{n}\right)=\left(\frac{\pi}{2} n\right)^{-2}
\end{gathered}
$$



Here the difference is that the real zeros are given for even $n$, and the imaginary zeros are given for odd n .

There are many more aspects to explore, like:
i. The coefficient 2 for the imaginary base zeros, and the coefficient $\pi / 2$ for the real base zeros, are related to $\lambda$ in $\lambda=\frac{2 \pi}{\ln \beta}$
ii. $\quad x_{0}$ for the imaginary bases is $\frac{1}{C}$ in $y+i z=i^{(c x)^{a}}$
iii. For $a=-2,-3$ and so on, there is an infinite wavelength with different asymptotic properties.
iv. For spirals with second level exponents like: $y+i z=(2 i)^{x^{\frac{1}{2}}}$ For positive $x$ the process of finding zeros is the same since only the amplitudes change. But there is a different structure for the zeros of the negative $x$ asymptotic spiral.


So, the basic principle is that the actual wavelengths are a function of the second level exponent. If the second level exponent is 1 , then there is just the usual wavelength. But, if the second level exponent is other than 1 , it modifies the usual wavelength in predictable fashion, even though the wavelengths themselves are varying.
$7.312^{\text {nd }}$ Level Exponents and the ' $x$-base'

## Recalling the $x$-base helix/spiral from section 6.6 The $x$ or Variable Base:

$$
y+i z=x^{i x}
$$



There is a technique from section 3.65 In Negative Real Base Form of making the base extremely large to greatly shorten the wavelength for easier viewing of the spiral's properties. So, adding a large coefficient and a second level exponent:

$$
y+i z=\left(\left(e^{6 \pi}\right) x\right)^{(i x)^{q}}
$$

the helix results for positive $x$, the spiral results for negative $x$, and with interval $-6 \leq x \leq 6$, the graph with $\mathrm{q}=1$ becomes:


And, the right end view:


The FRP real values in black, and the TIP imaginary values in red:


And, the three planar projections in 3d:


Following are the very interesting results occurring by making small changes to $q$ :
Violet = FRP, Red = TIP, Blue = RSIP



If ' $q$ ' modifies only $(x)$ rather than $(i x)$ in the exponent, i.e.:

$$
y+i z=\left(\left(e^{6 \pi}\right) x\right)^{i x^{q}}
$$

then the effect is to alter the amplitude of the negative ' $x$ ' part of the graph.
$q=.956$, interval $-6<x<6$ :

$q=.951$, interval $=-12<x<6:$

### 7.4 Exotic Graphs Using Second Level Exponents

## Example 1:

Combining a negative real base, a constant imaginary first level exponent, and a variable second level exponent that operates on $i$ only:

$$
y+i z=\beta^{b i^{x}}
$$

will generate Limacon/Cardioid Helixes of different shapes and sizes.

With $\beta=-0.39$ and $b=2$. (Note large amplitude scale):



With $\beta=1.74$ and $b=2.025$, a flattened Cardioid:



And with imaginary base, $\beta=i, b=1$ :

$$
y+i z=i^{i^{x}}
$$

FRP in Violet, TIP in Red, RSIP in Blue.



## Example 2:

Another simple approach is to take the basic $i$-base helix and use it both as base and exponent:

$$
y+i z=\left(i^{x}\right)^{\left(b i^{x}\right)}
$$

With $b=1$ and interval $-8 \leq x \leq 8$ :


And, with $b=-2$ :


## Example 3:

In a more complicated example, consider an $i$-base helix with a helix reciprocal sum exponent:

$$
y+i z=i^{\left(i^{x}\right)+\left(i^{x}\right)^{-1}}
$$

Notice that in the RSIP (blue) view the graph is a circle, but in the TIP (red) the graph is reversing direction every period. The full helix (black) shows how it can be seen either way. This is another example of how projections can leave out important information.



In an equation like this there are actually several potentially different numbers. There are the real and imaginary components of the first part of the exponent, the real and imaginary parts of the reciprocal part of the exponent, and the real and imaginary parts of the equation as whole.

If we use only the real component of the first part of the exponent, i.e.,

$$
y+i z=i^{r e a l}\left(i^{x}\right)+\left(i^{x}\right)^{-1}
$$

the graph looks like this:


Using the imaginary component,

$$
y+i z=i^{i m a g\left(i^{x}\right)+\left(i^{x}\right)^{-1}}
$$



Combining the different possible numbers in some parametric examples:

$$
\begin{gathered}
y=\operatorname{real} i^{\operatorname{real}\left(i^{x}\right)+\left(i^{x}\right)^{-1}} \\
i z=\operatorname{imag} i^{\operatorname{imag}\left(i^{x}\right)+\left(i^{x}\right)^{-1}}
\end{gathered}
$$

Notice that what can look like discontinuous 'cusps' in the projected side view are actually local maxima in the three dimensional helix view.


And, another exotic variation in two dimensions that is still a continuous three-dimensional helix. We are again choosing two of the possible numbers and graphing them against one another.

$$
\begin{gathered}
y=\operatorname{real} i^{\text {real }\left(i^{x}\right)+\left(i^{x}\right)^{-1}} \\
i z=\operatorname{imag} i^{\left(i^{x}\right)+\operatorname{imag}\left(i^{x}\right)^{-1}}
\end{gathered}
$$



And one more:

$$
\begin{gathered}
y=\operatorname{real} i^{\operatorname{real}\left(i^{x}\right)+\operatorname{real}\left(i^{x}\right)^{-1}} \\
\operatorname{iz}=\operatorname{imag} i^{\operatorname{real}\left(i^{x}\right)+\operatorname{imag}\left(i^{x}\right)^{-1}}
\end{gathered}
$$



## Example 4:

In this example of simply adding two helixes together with one having a different second level exponent, the result is a spiraling of the spiral:

$$
y+i z=\left(\left(i^{x}\right)+\left(i^{x^{a}}\right)\right)
$$

with orthogonal real (violet) and imaginary (red) components on the left, and helix (black) on the right,
and with $a=2.0032$ and interval $-4 \pi \leq x \leq 4 \pi$ :

and with $a=1.9972$ :


## Example 5:

In this example the base is the variable, the first level exponent is constant, and the second level exponent is also variable:

$$
y+i z=x^{i^{x}}
$$

The real values, FRP, in violet; and the imaginary values, TIP, in red; Interval $-12 \leq x \leq 12$ :



Animation 39 'Base and 2nd Level Exponent Variable'

## Example 6:

Here is a playful look at a variation of the equation from the end of section 7.31 2nd Level Exponents and the 'x-base': a function for a symmetrical castle tower which we stand on end (negative end of the horizontal axis goes vertical), and rotate:

$$
y+i z=\left(e^{6 \pi} x\right)^{a i x^{q}}
$$

With $a=3$ and $q=0.949$, and interval $-6 \leq x \leq 6$ :


[^5]
## 7.5 'Airy Type' Spiral

Second level exponents in a certain range on the usual exponential function will generate an asymptotic 'Airy Type' spiral. In other ranges for the second level exponent, helixes and expanding spirals result.

Beginning with an arbitrary value on a normal 2D exponential graph, for example:

$$
y=\operatorname{real} e^{(x)^{c}} c=1.11
$$

Graphing the FRP real values:


Although it is very difficult to see, the graph is periodic for negative x . The first two real zeros are at: -3.98318 and -10.71691 . And, the first two imaginary zeros are at -7.4375325 and -13.887596 .

To show the periodicity more clearly we simply raise the value of $c$ to $c=1.475$.
Real values in violet, imaginary values in red, and the RSIP side view in blue:


Animation 41 'Airy Type helix'

Adding a second term that is a reciprocal with various coefficients generates an Airy Type Spiral with Loops:

$$
\begin{gathered}
y+i z=A e^{a x^{s}}+B e^{-b t x^{s}} \\
s=1.574, a=8, b=-1, A=1, B=1.425
\end{gathered}
$$



### 7.51 Airy Zeros Approximations

Many reasonable approximations, to within about two or three significant digits, for Airy function zeros can be made in a general way with:

$$
y+i z=C \beta^{(a x)^{c}}
$$



Using the following coefficients an interesting first approximation to the Airy function zeros may be made:

$$
C=0.6149, \quad \beta=2.85, \quad a=1.2, \quad c=1.48
$$

What is particularly interesting about this function is that both sets of zeros from $\mathrm{Ai}(\mathrm{x})$ and $\mathrm{Bi}(\mathrm{x})$ are included in the real values.

FRP real values in blue, interval $-20 \leq x \leq 2$ :


| (b) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |


| Ai |  | -2.3381 |  | -4.0880 |  | -5.5206 |  | -6.7867 |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Bi | -1.1737 |  | -3.2711 |  | -4.8307 |  | -6.1699 |  | -7.3768 |
| helix | -1.0974 | -2.3053 | -3.2555 | -4.0866 | -4.8429 | -5.5461 | -6.2088 | -6.8391 | -7.4427 |

What this also tells us is that there is a whole second set of zeros for the imaginary values:


So, a second approach to approximating the Airy function zeros may be made by changing the base to $e$, finding new coefficients, and then combining the real values with the imaginary values by adding them and subtracting them.

If the two graphs are added together, then, for $R e+I m$ the following graph and approximations for $\mathrm{Bi}(\mathrm{x})$ result:

$$
\begin{gathered}
y=\operatorname{real} A e^{(a x)^{c}}+\operatorname{imag} B e^{(a x)^{c}} \\
A=.575, \quad B=.48 \\
a=.785, \quad c=1.48
\end{gathered}
$$



|  | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{Bi} \quad$ (b) | -1.1737 | -3.2711 | -4.8307 | -6.1699 | -7.3768 | -8.4920 | -9.5381 |
| Helix Re + Im | -1.1657 | -3.2639 | -4.8227 | -6.1668 | -7.3820 | -8.5076 | -9.5657 |

By adding a second pair of terms and adjusting coefficients, the first zero may be made as close as desired.

If the two graphs are subtracted from each other, then there are two options:

$$
R e-I m \text { and } \mathrm{Im}-\mathrm{Re}
$$

The approximations to the zeros are equally close, but center either on the nearer zeros or the further zeros. The following graph and approximations result; In this approach the zeros approximation to $\mathrm{Ai}(\mathrm{x})$ is not asymptotic for positive $x$ :

$$
\begin{gathered}
\operatorname{Ai}(x): \operatorname{Re}-\operatorname{Im}(\text { violet }) \quad y=\operatorname{real} A e^{(a x)^{b}}-\operatorname{imag} B e^{(a x)^{b}} \\
\operatorname{Ai}(x): \operatorname{Im}-\operatorname{Re}(\text { red }) \quad y=\operatorname{imag} A e^{(a x)^{b}}-\operatorname{real} B e^{(a x)^{b}} \\
A=.52, \quad B=.48 \\
a=.7775, \quad b=1.4832
\end{gathered}
$$



|  | 1 | 2 | 3 | 4 | 5 | 6 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Ai (b) | -2.3381 | -4.0880 | -5.5206 | -6.7867 | -7.9441 | -9.0227 |
| Helix Re - Im | -2.2680 | -4.0422 | -5.4923 | -6.7752 | -7.9497 | -9.0454 |
| Helix Im - Re | -2.3205 | -4.0820 | -5.5265 | -6.8063 | -7.9784 | -9.0724 |

## 7.6 'Bessel Type' Spiral

A slight variation of this approach allows for a 'Bessel Type' spiral. By adding the coefficient $i$ to the exponent:

$$
\begin{gathered}
y+i z=e^{(x)^{c}} \quad \text { Airy Type Spiral } \\
y+i z=e^{(i x)^{c}} \quad \text { Bessel Type Spiral }
\end{gathered}
$$

With $c=1.12$, the FRP (real values in violet), and TIP (imaginary values in red):



This helix, while having decaying amplitude values, does not have an asymptotic wavelength.
An additional example, with $c=2.9$ :


### 7.61 Bessel Zeros Approximation

The basic pattern for Bessel curves; $\mathrm{J}_{0}$ and $\mathrm{J}_{1}$ can be generated by the real and imaginary values respectively of a helix, with first and second level exponents as follows:

$$
y+i z=e^{f(i x)^{p}}
$$

$f=0.695$ and $p=1.1225$, Interval $-20 \leq x \leq 20$ :


This then gives for FRP:

$$
y=\operatorname{real} e^{f(i x)^{p}}
$$



With the following real values:

|  | 1 | 2 | 3 | 4 | 5 |
| :---: | :--- | :--- | :--- | :--- | :--- |
| $\mathrm{~J}_{0}$ (a) | 2.4048 | 5.5201 | 8.6357 | 11.7915 | 14.9309 |
| Helix Re | 2.1023 | 5.5943 | 8.8183 | 11.9005 | 14.8868 |A New Coordinate System for Complex Numbers360

And, in TIP:

$$
i z=\operatorname{imag} e^{f(i x)^{p}}
$$



With the following imaginary values:

|  | 1 | 2 | 3 | 4 | 5 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $J_{1}$ (a) | 3.8317 | 7.0156 | 10.1735 | 13.3237 | 16.4706 |
| Helix Im | 4.1434 | 7.2286 | 10.3735 | 13.4039 | 16.3517 |

For the first derivatives $J_{0}{ }_{0 \text { and }} J^{\prime}{ }_{1}$;

$$
y+i z=\frac{d}{d x}\left(e^{f(i x)^{p}}\right)
$$

$J_{0}$ approximated by the real values:

$$
y=\operatorname{real} \frac{d}{d x}\left(e^{f(i x)^{p}}\right)
$$

$J^{\prime}{ }_{1}$ approximated by the imaginary values:

$$
i z=\operatorname{imag} \frac{d}{d x}\left(e^{f(i x)^{p}}\right)
$$

The real values approximating $\mathrm{J}^{\prime}{ }_{0}$ in graph and table form:


|  | 1 | 2 | 3 | 4 | 5 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathrm{~J}_{0} \quad$ (a) | 3.8317 | 7.0156 | 10.1735 | 13.3237 | 16.4706 |
| Helix' Re | 3.6849 | 7.0310 | 10.1846 | 13.2209 | 16.1732 |

And, the imaginary values approximating $J^{\prime}{ }_{1}$ in graph and table form. This graph drops quickly near zero and is continuous at zero:


|  | 1 | 2 | 3 | 4 | 5 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathrm{~J}_{1}^{\prime}$ (a) | 1.8412 | 5.3314 | 8.5363 | 11.7060 | 14.8636 |
| Helix ‘ Im $^{2}$ | 1.8713 | 5.3904 | 8.6256 | 11.7149 | 14.7061 |

### 7.62 Variable Coefficient and Derivative Examples

Using an equation that has, a) the variable $x$ as a coefficient and b) certain larger second level exponents:

$$
\begin{gathered}
y+i z=x e^{(a+i x)^{c}} \\
a=0, c=2.995
\end{gathered}
$$

FRP real values:


With $a=1$, real values in violet, imaginary values in red:


With $x^{2}$ as a coefficient on a second derivative, and .25 scaling factor to make the amplitudes smaller for easier viewing:

$$
\begin{gathered}
y+i z=S x^{2} \frac{d^{2}}{d x^{2}}\left(e^{(a+i x)^{c}}\right) \\
a=1, \quad c=2.995, \quad S=.25
\end{gathered}
$$

FRP in violet, TIP in red:


Different exponents for $x$, including fractions and real numbers along with higher derivatives, modify the graph accordingly.

Multiple terms with different bases will also make corresponding modifications of the basic Bessel curves.

### 7.63 A Cornucopia Function

In another colorful example, if we use two terms of the Bessel Type with different coefficients, and then, add them and subtract them, as follows:

$$
\begin{gathered}
y+i z=\beta^{(b i x)^{c}}+\beta^{(g i x)^{h}} \\
y+i z=\beta^{(b i x)^{c}}-\beta^{(g i x)^{h}} \\
\beta=e^{p}, \quad p=.68, \quad b=1.375, \\
c=2.975, \quad g=1.375, \quad h=-1.05
\end{gathered}
$$

In two dimensions, real values for the sum (in violet), real values for the difference (in grey), interval $-6 \leq x \leq 6$ :


And, imaginary values for the sum (in red), imaginary values for the difference (in grey):


And in three dimensions, 'Four Cornucopias in 3Di':


Animation 42 'Cornucopia Helix Functions'

### 7.7 Higher Level Exponents

## $7.713^{\text {rd }}$ Level exponents - Two Examples

Example 1: 'A Hovering Arc Helix' and its 'Derivative Eight Helix'

Here is an equation similar to the example in section 7.2 Additional Levels of Exponents. This one has an asymptotic amplitude equal to $\beta^{c^{s}}$, and a very slowly decreasing wavelength:

$$
y+i z=\beta^{\left(c+i\left(\text { real } e^{(i t)^{s}}\right)\right)^{s}}
$$

With $\beta=e, c=2, s=1.044$, and interval $-20 \leq x \leq 20$, the RSIP in blue on the right appears as a simple arc. The corresponding helix is on the left.



The graph is periodic in both FRP (violet) and TIP (red), and appears to hover above the horizontal axis:


The first derivative of this equation:

$$
y+i z=\frac{d}{d x}\left[\beta^{\left(c+i\left(\text { real } e^{(i t)^{s}}\right)\right)^{s}}\right]
$$

generates an interesting Eight Helix. With $s=1$, the helix has constant amplitude.
$\beta=e, \quad c=2.5$


greg ehmka, 2013

With $s>1$ the amplitude decreases, and with $s<1$ the amplitude increases.

With $s=1.04$ and $\beta=e$, the Eight Helix has this decreasing amplitude. As $s$ moves further away from 1 additional shape changes occur.



## Example 2:

In an interesting variation of a hovering arc, the equation,

$$
y+i z=e^{i\left(e^{(b i x)^{s}}\right)}
$$

with $b=5, s=1.05$ ans interval $-3 \pi \leq x \leq 3 \pi$ gives the following:


Of the four possible numbers generated the real and imaginary parts of both the exponent and the whole term, parametrically graphing the following two:

$$
\begin{aligned}
& y=\operatorname{real} e^{i\left(\text { real } e^{(b i x)^{s}}\right)} \\
& i z=\operatorname{real} e^{i\left(\text { imag } e^{(b i x)^{s}}\right)}
\end{aligned}
$$

and then animating with $.994 \leq s \leq 1.05$, in RSIP projection, gives:


Animation 43 'Two Dimensional Radiated Wave'

## $7.724^{\text {th }}$ Level Exponent Example

Continuing the same idea of parametrically exploring two of the many possible numbers which may be generated, here is an example with fourth level exponents:

$$
\begin{gathered}
y+i z=e^{\left((\operatorname{dix})^{\left(e^{\left.(b i x)^{s}\right)}\right)}\right.} \\
\text { In violet, } y=\operatorname{real} e^{\left.\left((\operatorname{dix})^{(i m a g} e^{(b i x)^{s}}\right)\right)} \\
\text { In red, } y=\operatorname{imag} e^{\left((d i x)^{(i m a g} e^{\left.(b i x)^{s}\right)}\right)}
\end{gathered}
$$

With $d=1.2, b=0.5$, interval $-50 \leq x \leq 50$ :


And this three-dimensional graph which is using:

$$
y+i z=e^{\left((d i x)^{\text {real }\left(e^{(b i x)^{s}}\right)}\right)}
$$



Animation 44 '4th Level Exponent Helix'
greg ehmka, 2013

### 8.0 Helix and Spiral Antiderivatives

### 8.1 First Level Exponents

### 8.11 Helix Derivatives

The first four derivatives of the helix:

$$
y+i z=e^{i x}
$$

are:

$$
\begin{aligned}
\frac{d}{d x} e^{i x} & =i e^{i x} \quad(\text { violet }) \\
\frac{d^{2}}{d x^{2}} e^{i x} & =-e^{i x} \quad(\text { red }) \\
\frac{d^{3}}{d x^{3}} e^{i x} & =-i e^{i x} \quad(\text { blue }) \\
\frac{d^{4}}{d x^{4}} e^{i x} & =e^{i x} \quad(\text { black })
\end{aligned}
$$

These are equivalent to:

$$
\frac{d^{n}}{d x^{n}} e^{i x}=i^{n} e^{i x}
$$

and they are also equivalent to:

$$
\frac{d^{n}}{d x^{n}} e^{i x}=e^{i\left(x+\frac{n \pi}{2}\right)}
$$

Each $n$ represents a $\frac{\pi}{2}$ rotation that effectively phase shifts the helix. With interval: $0 \leq x \leq 4 \pi$ A

Since the phase shift $n$ can be any value, this is what allows for the concept of 'fractional derivatives'.

### 8.12 Helix Antiderivatives

In this 'phase shift' form:

$$
\frac{d^{n}}{d x^{n}} e^{i x}=i^{n} e^{i x}=e^{i\left(x+\frac{n \pi}{2}\right)}
$$

$n$ can also be negative. This results in a reverse rotation and therefore a negative phase shift which is the equivalent of an antiderivative of the helix. E.g. $n=-1$ :

$$
\frac{d^{-1}}{d x^{-1}} e^{i x}=i^{-1} e^{i x}=e^{i\left(x+\frac{-\pi}{2}\right)}=\int e^{i x} d x
$$

and:

$$
\iint e^{i x} d x=e^{i\left(x+\frac{-2 \pi}{2}\right)}, \quad \iiint e^{i x} d x=e^{i\left(x+\frac{-3 \pi}{2}\right)}
$$

As with the derivative and positive rotation/phase shift, the antiderivative and negative rotation/phase shift also allows for the concept of 'fractional antiderivatives'.

See section 9.31 The Polynomial Morphing Function, for a discussion of potential polynomial fractional derivatives.

### 8.2 First and Second level Exponents

### 8.21 First Level Exponent Imaginary:

One of the standard antiderivative forms for exponentials is:

$$
\begin{equation*}
\int x e^{c x^{2}} d x=\frac{1}{2 c} e^{c x^{2}} \tag{*}
\end{equation*}
$$

We can now extend this form to imaginary first level exponents, meaning to spirals and helixes:

$$
\int x e^{c i x^{2}} d x=\frac{1}{2 i c} e^{c i x^{2}}
$$

The derivative is on the left, and the antiderivative is on the right,
$y+i z=x e^{c i x^{2}}$

$$
y+i z=\frac{1}{2 i c} e^{c i x^{2}} \quad c=1
$$



### 8.22 Second Level Exponent Real

The constant second level exponent, 2 , in the previous example, can be extended to being a real number $(a+1)$ which can take on most any value; but, as a practical matter, at least for graphing, a value higher than 2 or 3 produces an increasing frequency extremely fast. So, we have:

$$
\begin{gathered}
\int x e^{c i x^{2}} d x=\frac{1}{2 i c} e^{c i x^{2}} \\
\int x^{a} e^{c i x^{a+1}} d x=\frac{1}{i c(a+1)} e^{c i x^{a+1}}
\end{gathered}
$$

For $c=1, a=.8$, the derivative is on the left and the antiderivative is on the right,


Below for $a$ negative, the derivative is, of course, not continuous at zero and positive $x$ increases upward, negative $x$ increases in the negative $i z$ direction. The antiderivative is continuous at zero.
$a=-.5, c=5 \quad$ ( $c$ changes the frequency):


### 8.23 Second Level Exponent Complex

The second level exponent may be made complex with the addition of an imaginary constant:

$$
\int x^{a+d i} e^{c i x^{a+1+d i}} d x=\frac{1}{i c(a+1+d i)} e^{c i x^{a+1+d i}}
$$

First, the antiderivative $y+i z=\frac{1}{i c(a+1+d i)} e^{c i x^{a+1+d i}}$ with
$a=1.95, c=1, d=0.215:$


[^6]And then, if this same equation is differentiated:
$y+i z=\frac{d}{d x}\left[\frac{1}{i c(a+1+d i)} e^{c i x^{a+1+d i}}\right]$


Animation 46 'Corresponding Helix Derivative'

### 8.24 First Level Exponent Complex

$$
\int x^{a} e^{(b+c i) x^{a+1}} d x=\frac{1}{(b+c i)(a+1)} e^{(b+c i) x^{a+1}}
$$

With the derivative on the left, and the antiderivative on the right, the coefficients are:
$a=0.25, b=1.625, c=2$


### 9.0 4Di: Function Morphing

### 9.1 4Di Function Input and Output alternatives

A 4Di morphing/animation function is of the form:

$$
y+i z=f(x, w)
$$

wherein ' $x$ ' is both input and output, whereas ' $w$ ' is input only and ungraphed.

In another parametric form, $x$ can be output only while $w$ is input only and ungraphed. E.g.

$$
\begin{array}{|l|}
\hline x=f(w) \\
y=g(w) \\
i z=h(w)
\end{array}=\begin{gathered}
x=3 \cos w \\
y=3 \sin w \\
i z=i \sin 3 w
\end{gathered}
$$

In this case, there is no animation even though there are still four variables. This is because $x$ plays only an output role.

This brings up the concept of whether $x$ (or any of several different possible variables) takes on an 'output only' roll or an 'input and output' role. For example, in section 4.65 Polynomial Nonlinearity, after applying the cubic formula, $x$ became output only while $y$ became both input and output. I.e.:

$$
x+i z=e^{\frac{2 \pi i k}{3}}(g(y))^{\frac{1}{3}} \text { for } k=1,2,3
$$

For any given variable, the possibilities may be:
I. input only and graphed
II. input only and ungraphed (parametric)
III. output and graphed
IV. output and ungraphed
V. input and output graphed or ungraphed

### 9.2 Morphing Concept

The concept of a 4Di 'morphing' is a certain kind of animation in which the various places that the morphing might pause or stop, can often be regarded as separate functions themselves in 3Di. In contrast, we might have something like a Motion or Displacement animation.

Technically, they are both just animated functions but the difference is in that the morphing changes the function's shape, whereas the displacement simply moves the graph around.

For example:

Morphing Function(in black): $y+i z=x^{w} \quad$ interval $2 \leq w \leq 3$

Displacement Function (in red): $\quad y+i z=x^{2}+i w \quad$ interval $2 \leq w \leq 3$

The Morphing Function changes the shape, in this case from a parabola to a cubic curve, whereas the Displacement Function changes the location, in this case, along the $i z$ axis:


### 9.3 4Di Morphing Function Examples

### 9.31 The Polynomial Morphing Function



## Animation 48 'Polynomial Morphing Function'

The animation above is of the equation:

$$
y+i z=x^{w} \text { interval: } 1 \leq w \leq 4
$$

As $w$ animates the graph and takes on integer values, the familiar straight line, parabola, cubic, and quartic result on the FRP. As $w$ takes on intermediate values, the negative part of the graph rotates around the $x$-axis.

Additional terms can represent the polynomial more fully. Even integers for $w$ produce 'even functions' and odd integers produce 'odd functions'.
E.g., with interval $3 \leq w \leq 6$ :

$$
y+z i=x^{w}-2 x^{w-1}-6 x^{w-2}
$$



## Animation 49 'Polynomial Morphing Function 2'

If one of the exponents becomes less than 1 , there will be a cusp at zero; and if one of the exponents becomes negative, there will be a discontinuity at zero.

### 9.32 Polynomial Fractional Derivatives

There is a possibility that the concept of fractional derivatives (section 8.11 Helix Derivatives) can also be applied to polynomials. In a typical polynomial:

$$
y+i z=x^{w}
$$

non-integer values for $w$ result in a partial rotation of the negative $x$ portion of the graph around the $x$-axis. See section 9.31 The Polynomial Morphing Function.

Then a usual polynomial first derivative would be:

$$
\frac{d}{d x} x^{w}=w x^{w-1}
$$

and a subsequent second derivative would be:

$$
\frac{d^{2}}{d x^{2}} x^{w}=w(w-1) x^{w-2}
$$

Speculating a fractional derivative with $n=1.8$, for example:

$$
\frac{d^{1.8}}{d x^{1.8}} x^{w}=C x^{w-1.8}
$$

the value of the coefficient $C$ might logically lie between the coefficients of the first and second derivatives. I.e.

$$
w<C<w(w-1)
$$

If a way can be found to interpolate the value of $C$ the notion of a fractional polynomial derivative defined in this way may be of use.

### 9.33 Exponential Morphing

$$
y+i z=e^{(i x)^{w}}
$$

With $-6 \leq x \leq 2$ and $1 \leq w \leq 3$, the graph shows changing shapes for negative $x$ as $w$ takes on certain values and ranges of values. Positive $x$ is always exponential. The pitch is continuously changing as well.
$w=1 \quad$ the usual exponential graph
$1<w<1.5$ asymptotic spiral
$w=1.5 \quad$ helix
$1.5<w<2$ expanding spiral, increasing amplitude
$w=2 \quad$ modified exponential
$2<w<2.5$ expanding spiral, decreasing amplitude
$w=2.5 \quad$ helix
$2.5<w<3$ asymptotic spiral
$w=3 \quad$ modified exponential


Animation 50 'Exponential Morphing’

### 9.34 Periodic Exponential Morphing

By adding the coefficient $i$ to the exponent, and placing an animation variable $w$ as an exponent to $i$, the graph will be periodic. As $w$ takes on the integer values $0,1,2,3 \ldots ., i^{w}$ will take on the values $1, i,-1,-i$, and repeat. And then, in this case, the graph will cycle from the exponential, to the helix, to the negative exponential, to the negative helix, and repeat. Intermediate values for $w$ will result in spirals for the graph.

$$
y+i z=e^{i^{w} x}
$$

Intervals $-8 \leq x \leq 8$ and $0 \leq w \leq 8$ :


Animation 51 'Periodic Exponential Morphing'

### 9.35 Parabolic Morphing

If an animation variable $w$ is placed as the exponent on a parabola, e.g.,

$$
y+i z=\left(x^{2}+x-1\right)^{w}
$$

the vertex of the parabola will rotate and change shape. For $0 \leq w \leq 1$, conic nonlinear roots graphs will appear; and, for negative $w$, familiar inverse graphs will appear with the center segment rotating inside the discontinuities.

With intervals $-4 \leq x \leq 4$ and $-3 \leq w \leq 6$ :


Animation 52 'Parabolic Morphing'

### 9.36 Lissajous Morphing

In a variation of the two-dimensional Lissajous curve, rather than graphing two different frequencies perpendicular to each other, the real values of a function and the imaginary values of its reciprocal function are graphed perpendicular to one another. The frequencies will be the same but, the amplitudes, phase relationship, and development (CCW for the function and CW for its reciprocal) will be different. And then, the animation variable changes the frequency for both.

$$
\begin{gathered}
y+i z=\left[\operatorname{real}\left(e^{w\left(x^{c}(n+i)\right)^{q}}\right)\right]+i\left[\operatorname{imag}\left(e^{w\left(x^{c}(n+i)\right)^{q}}\right)^{-1}\right] \\
c=.493, n=.9773, q=2
\end{gathered}
$$

The coefficient $i$ is necessary for the second term since taking the imaginary value of a function converts it to a real number. This allows the expression to be in one equation. Alternatively, $y$ and $z$ can be expressed parametrically.

The equation graphs a helix with changing 'elliptical eccentricity' as a function of changing frequency. The reciprocal nature of the two terms graphed perpendicular to each other, determines the changing eccentricity as one gets large and the other gets small. In two dimensions, there is a resemblance to a galactic disk.

Intervals $-10 \leq x \leq 10$ and $0 \leq w \leq 6$ :


Animation 53 'Lissajous Morphing 2D’

In three dimensions the increasing 'elliptic eccentricity' of the helix is shown clearly:


## Animation 54 'Lissajous Morphing 3Di'

## $9.372^{\text {nd }}$ Derivative Morphing

In a more complicated example, here is an interesting application of the morphing idea. The equation is the second derivative of an 'Airy Type' helix, ostensibly occurring over time. The graph appears to model in successive stages, turbulence, projectile resistance, and impact. What is most interesting, is that the region of highest amplitude of the wave is in the middle, and moves forward.

For $t=.25$, the amplitude is near zero, around $x=2$; grows to a high of around 13, around $x=750$; and is below 1 again, at $x=8000$. At $t=1$, the rise and fall of the amplitude occurs below $x=500$. And at $t=3$, below $x=55$.

$$
\begin{gathered}
y+i z=\frac{d^{2}}{d x}\left(e^{.18 t\left(3.2 x+\frac{2 i}{3}\right)^{\frac{3}{2}}}\right. \\
0 \leq t \leq 9
\end{gathered}
$$



## Animation 55 'Second Derivative Morphing 2D'

And in three dimensions: (The graphing software is a little challenged by this one; so, the $x$ interval is shortened.)
Interval $-23 \leq x \leq 0, t$ is still $0 \leq t \leq 9$ :

## Animation 56 'Second Derivative Morphing 3Di'

### 9.38 Morphing Skewed Spirals

Spirals can be skewed and morphed using multiple terms, different exponents or derivatives. A typical equiangular spiral using the $i$ rotator from section 9.39 A Simple Rotator:

$$
y+i z=e^{x i^{w}} \quad w=.85
$$

In RSIP with interval $-20 \leq x \leq 50$ :


This can be skewed with:

$$
y+i z=e^{x i^{w}}+e^{x i^{w-c}} \quad w=.85, \quad c=1.8
$$



And/or further skewed with derivatives:

$$
y+i z=\frac{d^{n}}{d x^{n}}\left(e^{x i^{w}}+e^{x i^{w-c}}\right) n=1,2,3
$$



And, then morphed with $.8 \leq w \leq .9$ :


Animation 57 'Morphing Skewed Spirals 2D'


Animation 58 'Morphing Skewed Spirals 3Di'
greg ehmka, 2013

### 9.39 A Simple Rotator

Similar to section 9.34 Periodic Exponential Morphing, where periodic morphing of the $e$-base results from using $i^{w}$ as a coefficient in the exponent, this 'periodic coefficient' can be used to rotate any function around the $x$ axis.

Taking a somewhat exotic curve in 3Di side view projection:

$$
y+z i=i^{a i^{b i}}
$$

With $a=3, b=2,0 \leq x \leq 2$ :

and, adding the 'periodic coefficient', making it 4Di:

$$
y+z i=i^{w} i^{a i^{b i}}
$$

allows the curve to continuously rotate or pause anywhere depending on $w$.

Here, the interval is: $0 \leq w \leq 20$ :


Animation 59 'Simple Periodic Rotation Coefficient'

Notice that here, in the above graph, the periodic coefficient, or simple rotator, is used as both coefficient and as second level exponent. This means it can be used as an 'animator', with an animation variable $w$ or to determine a circular geometry with $x$ as input.

This periodic coefficient, or simple rotator, with animation variable $w$, i.e. $i^{w}$, is an interesting entity - almost acting as an operator. It is, of course, equivalent to $e^{\frac{w \pi i}{2}}$, but, somewhat easier to work with. Used as a coefficient on a function, it can provide rotation around the $x$ axis or phase shift (black object). And, used as an exponent, can provide function morphing: in this case, between the exponential graph and the helix graph with equiangular spirals in between (red object).

$$
\begin{gathered}
y+i^{w} e^{i x} \\
\text { phase shifting helix in black }
\end{gathered}
$$

$$
y+i z=e^{x i^{w}}
$$

Exp/Helix/Spiral morphing in red

Intervals $-2 p \leq x \leq 2 p i$ and $0 \leq w \leq 8$ :


## Animation 60 'Simple Periodic Rotation Coefficient 2'

When the Periodic Coefficient is used with a constant, it provides simple circular rotation in the RSIP.

$$
\begin{gathered}
y+i z=i^{x} 4 \quad 2 D \text { orbit in black } \\
y+i z=i^{w} 4 \text { rotating object in red }
\end{gathered}
$$

(The object being rotated is a default object supplied by the software.) After Closed Surfaces are covered in section 12.2 Closed Surfaces, the objects themselves can become part of the equation.


Animation 61 'Simple Periodic Rotation Coefficient with a Constant

### 9.40 The Equiangular Spiral and Cardioid Motion

When the periodic coefficient is used as an exponent, a Cardioid Type orbit and motion result:

$$
\begin{aligned}
& y+i z=4^{i^{x}} \text { Cardioid type orbit in black } \\
& y+i z=4^{i^{w}} \text { Cardioid type motion in red }
\end{aligned}
$$

In RSIP:


The object that we are calling a 'Cardioid' (black), isn't quite a cardioid (red), and not quite a Cardioid Petal (violet), and not quite Cayley's sextic (blue).


It has, however an extraordinary property; that being, it is the path of a point on an equiangular spiral as the spiral morphs.

If we take an $x$-axis interval, $0 \leq x \leq 4 \pi$, and choose several points, $P_{n}$, on that interval. And then, graph this type of Cardioid for each point:

$$
y+i z=P_{n}^{i^{x}} \text { for } n=.1, .5,3,4,6,11
$$

in two dimensions, RSIP, we have:


If we then add the equiangular spiral with a specific value for $w$ :

$$
y+i z=x^{i^{w}} \quad w=2.37
$$



The points of intersection between the equiangular spiral and the Cardioid type orbits will follow the Cardioid orbits. This form of equiangular spiral is not continuous at zero.

So, with interval $0<w \leq 4$ :


Animation 62 'Equiangular Spiral with Cardioid Orbits 2D'

Notice that the points which are inverses, i.e. $n=.1, .5$, orbit half a cycle ( 2 in the $i$ cycle of 4 ) out of phase with the integer points.

In 3Di:


Animation 63 'Equiangular Spiral with Cardioid Orbits 3Di'

These constructions are with $x^{i^{w}}$ which is not continuous at zero for all values of $w$, but in two dimensions is the same equiangular spiral as $e^{x i^{w}}$ which is continuous at zero for all values of $w$. If, alternatively, $e^{x i^{w}}$ is used, the exact same orbits will occur except at different locations, i.e., different values for $x$. Additionally, the two different types of equiangular spiral in three dimensions have very different ways that the wavelength varies.

### 10.0 4Dii: Helix Morphing

In 4Di morphing there are three real variables and one imaginary variable:

$$
y+i z=f(x, w)
$$

The morphing variable can also be imaginary:

$$
y+i z=f(x, i w)
$$

in which case the coordinates become 4Dii: two real variables and two imaginary variables. There are different forms of 4Dii. See section 11.1 Functions in 4Dii, for discussion.

The black helix function below is:

$$
y+i z=\pi^{x i c}+\pi^{x i w}
$$

with: $c=2$ and intervals:

$$
-36 \leq x \leq 26 \text { and }-2 \leq w \leq 3
$$

The blue object is the side view (RSIP) of the helix. And, the red object (above the blue object) is the side view of the resulting helix after performing the following integral operation on the helix:

$$
y+i z=\int_{0}^{x}\left(\pi^{x i c}+\pi^{x i w}\right) d x
$$

Notice, in general that the blue object has loops whereas the red object has cusps. Also notice, at various places, the associated curves (evolute, pedal, etc.), show relationships between the two. For example, the one shown in the starting picture of the Tricuspoid and Trifolium for $w=-1$ :


Animation 64 'Helix Morphing'

### 11.0 4Dii: Algebraic Forms

### 11.1 Functions in 4Dii

The types of functions covered so far have been the following:
i. 3 Di - two real variables and one imaginary variable:

$$
y+i z=f(x)
$$

ii. 4Di morphing - three real variables and one imaginary variable:

$$
y+i z=f(x, w)
$$

iii. 4Dii morphing - two real variables and two imaginary variables:

$$
y+i z=f(x, i w)
$$

In this section we consider two more versions of 4Dii.
i. $\quad y+i z=f(x+i \theta)$
ii. $\quad y+i z=f(u+i v)$

The first has an algebraic relationship between the two input variables. It is that of a complex number where the imaginary part of the input is an angle. This means that there is a single complex number as input and a single complex number as output. This will be referred to as line-angle input.

The second also has an algebraic relationship between the two variables. But, in this case, the input is that of a 'complex regional domain', rather than the line-angle input of the first one. A complex regional domain is essentially a rectangular complex area with length $u$ and width $i v$, or the reverse, length $i v$ and width $u$. See section 12.4 Complex Regional Input

Also see section 13.1 Observable and Embedded Dimensions, for further discussion of function types and input alternatives.

### 11.2 A New Geometry of Natural logarithms

### 11.21 Spira Mirabilis Again

If the usual Euler Helix:

$$
y+i z=e^{i x} \text { in black }
$$

has an additional real coefficient in the exponent:

$$
y+i z=e^{a+i x} \text { in red, } a=1
$$

the result is an exponential increase in the helix's amplitude:


And, if the constant $a$ and the variable $x$ exchange places:

$$
y+i z=e^{x+i a} \quad a=1
$$

the result is a specific state of rotation of the exponential graph:


Combining the helix with the exponential, meaning the variable $x$ is applied to both parts of the exponent, e.g. the complex number $(1+3 i)$ :

$$
y+i z=e^{x(1+a i)} \quad a=3
$$

results in-the exponentially spiraling helix that is the Complex Exponential form for the equiangular spiral from section 3.62 The 'Complex Exponential' Form The curvature is determined by $a$ :


### 11.22 The Four Coordinate Complex Exp/Log Function

The above equiangular spiral uses the same variable $x$ on both the real part and the imaginary part of the exponent. If we let $x$ be the real part, and the imaginary rotational part be $\theta$ :

$$
y+i z=e^{x+i \theta}
$$

then, this exponential function and its inverse, the complex logarithmic function:

$$
\ln (y+i z)=x+i \theta
$$

will be seen to have the striking and wonderful relationship of a fully correlated four coordinate system!

$$
(x, y, i z, i \theta)
$$

In which:

- The real input $x$ is on the horizontal axis.
- The imaginary input $i \theta$ will be the specific rotation; the value of which is the rotation of the graph.
- The real output $y$ is on the vertical axis.
- And the imaginary output is on the depth axis.

A point on this graph with input $(1+i)$ :

will have the coordinates:

$$
(x, y, i z, i \theta)=(1,1.46869,2.28736 i, i)
$$

related by:

$$
y+i z=e^{x+\theta i}
$$

or:

$$
\ln (y+i z)=x+\theta i
$$

as will be the case for any point on the graph with any rotation. The fourth coordinate i $i \theta$ provides the means whereby the equation remains a function, even with two or more points on the same vertical line - which would ordinarily disqualify the equation as a function.

Additionally, the rotations of the graph still make the natural log function multi-valued, so, considerations relative to Principal Value are still needed.

### 11.23 The Four Coordinate Complex Exp/Log Surface

Further, using the version of 4Dii:

$$
y+i z=f(u+i v)
$$

provides a complex regional domain as input rather than the line-angle input. This form of input allows the function to have a surface by, in effect, graphing all values of the rotation at once.

In this case the coordinates are:

$$
(x, y, i z, i \theta) \rightarrow(u, y, i z, i v)
$$

So:

$$
y+i z=e^{u+i v}
$$

and:

$$
\ln (y+i z)=u+i v
$$

with Principal Value considerations still playing a role for values of $i v$.

Graphing $u$ on the horizontal and intervals, $-2 \pi \leq u \leq 1.5$ and $0 \leq v \leq 2 \pi$, results in the surface:


Note that in this context Euler's 'famous five' identity:

$$
e^{i \pi}+1=0
$$

specifies a point in 4Dii coordinates! Meaning that if we define four observable dimensions as horizontal, vertical, depth, rotation, then:

$$
(h, v, d, r)=(x, y, i z, i \theta)
$$

and if we slightly rearrange Euler's identity to:

$$
-1+i 0=e^{0+i \pi}
$$

then it will be in the four coordinate form:

$$
y+i z=e^{x+i \theta}
$$

and reduce to:

$$
-1=e^{i \theta}
$$

and so the four coordinates become:

$$
(h, v, d, r)=(x, y, i z, i \theta)=(0,-1, i 0, i \pi)
$$

and specify a point, (red ball), on the Exp/Ln surface:


Going back to the surface itself, $y$ and $i z$ are graphed along the vertical and depth axes respectively. But, there is an option as to whether these two outputs are graphed against the real or the imaginary part of the input $u$ or $v$ on the horizontal $x$ axis. Graphing them against $u$ results in the surface object above. Meaning:

$$
\begin{gathered}
x=\operatorname{real}(u+i v) \\
y+i z=e^{u+i v}
\end{gathered}
$$

Graphing them against $v$, the imaginary part of $(u+i v)$, results in a different surface object. See section 13.0 4Dii: Object-Wave Duality Surfaces. Also see section 12.3 Open Surfaces for more open surfaces.

### 11.3 The Imaginary Logarithmic Surface

As noted in section 6.51 The $i$-base Exponential, the $i$-base exponential graph, $i^{i z}$, also has a rotation associated with it when given a complex exponent, $\theta$, which in that case is real.

Also, in section 6.44 Imaginary Wavelength Interpretation, the $i$-base helix, $i^{x}$, was seen to have an imaginary wavelength. Bringing these together, the $i$-base exponential graph with a complex exponent was defined to have an orientation along the depth axis and a rotation that is real. This suggests the same modification to 4Dii as we made there to 3Di. If the $i$-base exponential graph with rotation, meaning having a complex exponent, is oriented along the depth axis, then the imaginary output must fall along the $x$ axis. Then, as in section 6.52 Rotating Exponentials:

$$
y+i x=i^{\theta+i z}
$$



## Animation 65 'i-base Exponential Rotation'

This says that the complex output has real part, which is graphed on vertical ' $y$ '; and imaginary part, which is graphed on the horizontal ' $x$ ', which makes $x$ imaginary. Then:

$$
y+i x=i^{\theta+i z}
$$

provides imaginary logarithms by the change of base equation:

$$
l i(y+i x)=\log _{i}(y+i x)=\frac{\ln (\mathrm{y}+\mathrm{ix})}{\ln i}=\theta+i z
$$

And then, there is a four coordinate function just as with the natural logarithms, but, with the following changes:

$$
\begin{aligned}
& (\text { hor , vert, depth, angle })=(x, y . i z, i \theta) \text { for e base } \\
& (\text { hor, vert, depth, angle })=(i x, y . i z, \theta) \text { for } i \text { base }
\end{aligned}
$$

Switching to ( $u+i v$ ), complex regional input for the surfaces, the $e$ base logarithmic surface uses $u$ for the horizontal, or $x$ axis, and is oriented along the $x$ axis, meaning that the horizontal axis is the input. The $i$ base logarithmic surface, in contrast, is oriented along the depth or $i z$ axis, meaning that the depth axis is the input.
$e$ base:

$$
\begin{gathered}
x=u \\
y+i z=e^{u+i v}
\end{gathered}
$$

$i$ base:

$$
\begin{gathered}
z=v \\
y+i x=i^{u+i v}
\end{gathered}
$$

And so, with intervals $0 \leq u \leq 4$ and $-.5 \leq v \leq 4$, the imaginary logarithmic surface is:


And, any point on the surface has coordinates:

$$
(\text { hor, vert, depth, angle })=(i x, y, i v, u)
$$

related by:

$$
y+i x=i^{u+i v}
$$

or:

$$
l i(y+i x)=u+i v
$$

Noting again that for the $i$ base, the real part of the exponent is the rotation. With $\theta$ using lineangle input, or with $u$ using complex regional input, imaginary logs (like natural logs) are multivalued. The period of rotation for the $i$ base is 4 . So, if $\theta$ or $u$ are greater than 4 , the same Principal Value considerations as with natural logs are needed.

As with the $e$ base, there is an option to graph the output with either of the input variables along the $i z$ axis. This will be explored further in section 13.0 4Dii: Object-Wave Duality Surfaces.

### 11.4 More Exponential/Logarithmic Surfaces

The concept of logarithmic surfaces can be extended to a wide range of bases and functions in the exponents. In a general way, these can be expressed in either exponential form or logarithmic form. I.e.:

$$
\begin{gathered}
y+i z=\beta^{f(u)+g(i v)} \\
\log _{\beta}(y+i z)=\frac{\ln (y+i z)}{\ln \beta}=f(u)+g(i v)
\end{gathered}
$$

And then, making provision for whether the base is real or imaginary and is therefore oriented along the $x$ or $i z$ axis as well as considering Principal Values.

## Example One:

As we've seen, the imaginary and natural logarithmic surfaces are asymptotic to the input axis. Here is an $i$ base example of a surface being asymptotic to a cylinder:

$$
y+i x=i^{u+i^{i v-1}}
$$

The depth input axis is graphed with $v$ values and the intervals are:
$0 \leq u \leq 4$ and $0 \leq v \leq 10$


## Example Two:

A complex base example:

$$
y+i z=(e+i)^{u+v i}
$$

And, therefore:

$$
\log _{(e+i)}(y+i z)=\frac{\ln (y+i z)}{\ln (e+i)}=u+v i
$$

Input $u$ values along the horizontal. Intervals $-2 \pi \leq u \leq 1.5$ and $0 \leq v \leq 2 \pi$ :


The 'break' in the graph is a morph of the exp/natural log surface graph depending on the value of $a$ :

$$
y+i z=(e+a i)^{u+v i} \quad a=1
$$

## Example Three:

The concept will also work for variable bases. I.e.:

$$
y+i z=u^{u+v i}
$$

and, ostensibly:

$$
\log _{u}(y+i z)=\frac{\ln (y+i z)}{\ln u}=u+v i
$$

But, as might be expected, the period moves around with the value of $u$, and so, there will be varying principal values.

Input: $v$ values on the horizontal. Intervals $0 \leq u \leq 2$ and $-10 \leq v \leq 25$ :


### 12.0 4Dii: Surfaces

### 12.1 Surfacing a Function

In the previous examples of exponential/logarithmic functions, using line-angle input:

$$
y+i z=e^{x+i \theta}
$$

the angle part of the input provides a rotation of the graph. If used with a second level exponent there is, in addition, a morphing of the function. For example:

$$
y+i z=e^{(x+i \theta)^{c}} \quad c=1.525
$$

Intervals $-2 \pi \leq x \leq 4.5$ and $0 \leq \theta \leq 2 \pi$ :


## Animation 66 'Morphing and Rotating an Exponential'

Switching to complex regional domain input with the same equation:

$$
y+i z=e^{(u+i v)^{c}} \quad c=1.525
$$

With intervals, $-2 \pi \leq u \leq 1.5$ and $0 \leq v \leq 2 \pi$, the variable $v$, in effect, graphs all of the values of $\theta$ at once, and the following surface results:


The surfacing operator $e^{i v}$ can be applied to other functions. I.e.:

$$
y+i z=e^{i v} f(u)
$$

For example, if the function is a polynomial:

$$
f(u)=u^{3}+u^{2}-u
$$



Applying the surface operator:

$$
y+i z=e^{i v}\left(u^{3}+u^{2}-u\right)
$$

Intervals $-2 \leq u \leq 2$ and $0 \leq v \leq 2 \pi$ :


### 12.11 Inverse Functions

Finding $u$ and $v$ with inverse functions is possible in many cases, generally by taking logs of both sides, setting real and imaginary parts equal to one another, and solving for $v$ and $u$ where possible:

$$
\begin{gathered}
y+i z=e^{i v} f(u) \\
\ln (y+i z)=i v+\ln (f(u)) \\
v=\operatorname{imag} \ln (y+i z) \\
f(u)=e^{\text {real }(\ln (y+i z))}
\end{gathered}
$$

The steps for the sphere are in the next section.

### 12.2 Closed Surfaces

### 12.21 The Sphere Surface Function

Returning to our original conic from section 2.6 Conics in 3Di, and this time arranging it so that the circle falls on the FRP, rather than the hyperbola; and then, solving for $y$ :

$$
y= \pm\left(1-x^{2}\right)^{\frac{1}{2}}
$$


taking just the upper half circle function, and, adjusting the interval to include only the circle portion and not the hyperbola portion:

replacing $x$ with $u$ :

$$
f(u)=\left(1-u^{2}\right)^{\frac{1}{2}}
$$

and, applying the surface operator:

$$
\begin{gathered}
y+i z=e^{i v} f(u) \\
y+i z=e^{i v}\left(1-u^{2}\right)^{\frac{1}{2}}
\end{gathered}
$$

results in a four coordinate function $(u, y, i z, i v)$ for the surface of a sphere!

Intervals $-1 \leq u \leq 1$ and $0 \leq v \leq 2 \pi$ :


To find the sphere surface inverse functions:
i. take the natural logs of both sides,
ii. set the real and imaginary parts equal to each other,
iii. and, solve for $u$ and $v$.

$$
\begin{gathered}
y+i z=e^{i v}\left(1-u^{2}\right)^{\frac{1}{2}} \\
\ln (y+i z)=i v+\ln \left(\left(1-u^{2}\right)^{\frac{1}{2}}\right) \\
v=i m a g \ln (y+i z) \\
\ln \left(\left(1-u^{2}\right)^{\frac{1}{2}}\right)=\text { real } \ln (y+i z)
\end{gathered}
$$

Due to the symmetry of the sphere, for a given $(y+i z)$, there will be two values of $u$. And, these are given by the two square roots:

$$
u= \pm\left((-1)\left(e^{2(\operatorname{real}(\ln (y+i z)))}-1\right)\right)^{\frac{1}{2}}
$$

For a radius of the sphere other than 1 :

$$
\begin{gathered}
y+i z=e^{v i}\left(r^{2}-u^{2}\right)^{\frac{1}{2}} \\
u= \pm\left((-1)\left(e^{2(\text { real }(\ln (y+i z)))}-r^{2}\right)\right)^{\frac{1}{2}}
\end{gathered}
$$

To include only the sphere, the interval for $u$ should be:

$$
-r \leq u \leq r
$$

Extending the interval for $u$ outside the boundary of the sphere results in hyperboloid extensions. With $-3 \leq u \leq 3$ :


To find inverses on these extensions we would begin with the equation of the hyperbola, rather than the equation of the ellipse. I.e.:

$$
y= \pm\left(x^{2}-1\right)^{\frac{1}{2}}
$$

In effect, beginning with either the ellipse equation or the hyperbola equation specifies whether the ellipse is 'real' and the hyperbola 'imaginary', or the reverse.

### 12.22 The Cube Surface Function

If we take the geometry function, $G$, of the half circle, above, and raise the exponent, i.e.:

$$
G=\left(1-u^{100}\right)^{\frac{1}{2}}
$$

the half circle approaches the shape of a square:


And then, set a surface operator, $S$, such that it generates a square:

$$
\begin{gathered}
S=A e^{a i v}+B e^{-b i v} \\
A=-15, B=1.125, a=3, b=1, a+b=4
\end{gathered}
$$



Combining the geometry function, $G$, with the surface operator function, $S$, will result in a surface function for a cube:

$$
\begin{gathered}
y+i z=S G \\
-1 \leq u \leq 1 \text { and } 0 \leq v \leq 2 \pi
\end{gathered}
$$



Technically, the object is more of a 'cube lozenge' since the corners and edges will always have some tiny amount of curve to them and the end surfaces will always have some tiny amount of slope to them. But, with refinements, it likely can be made as close as desired.

### 12.23 Lozenges, Barrels and Pointed Cylinders

## "Chubby Pentagon Lozenge"



Using the same principle as with helix functions we can define a 'surface operator', $S$, to have more than one term - for example, a reciprocal sum:

$$
\begin{gathered}
S=A e^{a i v}+B e^{-b i v} \\
0 \leq v \leq 2 \pi
\end{gathered}
$$

Applying this to the equation of a half circle:

$$
y+i z=S\left(1-u^{2}\right)^{\frac{1}{2}}
$$

results in a vast array of possible closed surfaces. In a general way, the integers, $a$ and $b$, determine the number of cusps or loops, and $A$ and $B$ determine the shape of the cusps or loops. E.g., with $A=3, B=.97, a=1, b=2$ :


Animation 67 'Triangular Lozenge'

With $A=3, B=1.65, a=1, b=5$ :


And, with $A=.15, B=3, a=5, b=1$ (Hexagon 'Saturn' Lozenge):


Additionally, if we define $f(u)$ to be the 'geometry function', $G$, changes to $G$ add further possibilities. For example, if the second exponent $q$ takes on different values:

$$
\begin{gathered}
G=\left(1-u^{2}\right)^{q} \\
y+i z=S G
\end{gathered}
$$

$S$ is the surface operator function, and $G$ is the 'geometry function'.

With $a$ and $b$ equal, $A=3, B=1.325$ and $q=.05$, an elliptical lozenge with sharper boundaries results:


Further possibilities result by varying either the constant, $r$, or the first exponent, $p$ :

$$
G=\left(r^{2}-u^{p}\right)^{q}
$$

Keeping the other coefficients the same, with: $r=1$ and $p=1.975$, the lozenge opens on one side:


And, with $r=1.025$ and $p=2$, the lozenge opens on both sides:


Adjusting $q$ determines the size of the opening. With $q$ back to . 5:


Adding another cusp with $a=2$, and adjusting $A$ and $B$ results in a triangular hole in a triangular lozenge:


Going back to the sphere, with $A=1, B=0, a=1$ :


And, closing the hole with $r=1$. Then, changing the second exponent to $q=.025$, we have a closed ended barrel:


Changing the horizontal interval to $-.7 \leq u \leq .7$, and matching $r$ to it with $r=.7$, and then with $q=3$, we have a pointed cylinder:


### 12.3 Open Surfaces

### 12.31 Constant, Exponential and Polynomial Surfaces

What we are calling the surface operator function, $S$ :

$$
S=A e^{a i v}+B e^{-b i v}
$$

can be applied to many different 'geometry' functions, $G$. If $G$ is a constant:

$$
G=1
$$

a simple cylinder results that can be made to have various shapes.

With intervals $-6 \leq u \leq 6$, and $0 \leq v \leq 2 \pi$ and $A=.9, B=6, a=4, b=1, a+b=5$ :


If $G$ is the exponential function, it can also be made to have different shapes:

$$
G=e^{u}
$$

```
Intervals -8 \lequ\leq1.5 and 0\leqv\leq2\pi
and A=.475, B=3,a=3,b=1,a+b=4:
```



Going back to line-angle input on an exponential graph:

$$
y+i z=\left(A e^{a i \theta}+B e^{-b i \theta}\right) e^{x}
$$

With the same coefficients as for the surface, the rotating exponential graph will outline the square surface. The movement of this square rotating exponential is interesting.

In addition to the path of a point (the grey ball on the exponential graph that will trace out a square) the actual rotation of the graph undergoes changes in its rotational velocity. It decelerates at the corners of the square and accelerates along the sides of the square.


Animation 68 'Square Rotating Exponential'
$G$ can also be a polynomial. For a typical cubic:

$$
G=\left(u^{3}+u^{2}-u-1\right)
$$

Interval $-2 \leq u \leq 2$ :


For just a simple circular surface:

$$
S=A e^{a i v}+B e^{-b i v}
$$

greg ehmka, 2013


And, a more complicated 'Cardioid Limacon Cubic Surface' still using:

$$
\begin{gathered}
G=\left(u^{3}+u^{2}-u-1\right) \\
S=A e^{a i v}+B e^{-b i v} \\
y+i z=S G
\end{gathered}
$$

$A=1.575, B=1, a=1, b=2:$


Animation 69 'Cardioid Limacon Cubic Surface'

An exponential function can be added to the cubic:

$$
G=e^{u}\left(u^{3}+u^{2}-u-1\right)
$$

Intervals $-12 \leq u \leq 1.3$ and $0 \leq v \leq 2 \pi$
and $A=1.575, B=1.65, a=5, b=2, a+b=7$ :


Animation 70 'Seven Loop Exponential Cubic Surface'

### 12.4 Complex Regional Input

What we are calling a complex regional domain is a simple complex rectangular area with dimensions $u$ and iv (violet).

$$
x+i z=u+i v
$$

This area changes shape with an exponent, e.g., $a=2$ (blue), and $a=3$ (yellow). And, for various exponents assumes various sizes and shapes (red animation).

$$
x+i z=(u+i v)^{a}
$$

In two dimensions, with intervals $-1 \leq u \leq 1,-1 \leq v \leq 1,-8 \leq a \leq 8$ :
D)


Animation 71 'Simple Complex Area'

In three dimensions, for example, graphing both of the square root functions of a complex variable:

$$
\begin{gathered}
x=u \\
y+i z=(u+i v)^{\frac{1}{2}}
\end{gathered}
$$

Interval to $-2 \leq u=v \leq 2$ :


### 12.41 Cornu Spiral/Fresnel Surface

A flat or linear surface, rather than a circular one, is also possible. Using a Fresnel Integral, with the $v$ interval as the limits of integration:

$$
\begin{gathered}
x=u \\
y+i z=\int_{-v}^{v}\left(e^{u^{2} \frac{i \pi}{2}}\right) d u
\end{gathered}
$$

Intervals $-1.5 \leq u \leq 1.5$ and $-\pi \leq v \leq \pi$ :


Going back to a circular surface, with $u$ as the limits of integration:

$$
\begin{gathered}
y+i z=e^{i v} \int_{-u}^{u}\left(e^{u^{2} \frac{i \pi}{2}}\right) d u \\
-6 \leq u \leq 6 \text { and } 0 \leq v \leq 2 \pi
\end{gathered}
$$



Placing the surfacing operator itself as the limit of integration generates a cylindrical Nephroid surface.

$$
\begin{gathered}
y+i z=\int_{0}^{e^{i v}}\left(e^{u^{2} \frac{i \pi}{2}}\right) d u \\
-3 \leq u \leq 3
\end{gathered}
$$



Modifying the exponent on $u$ will determine the number of 'circular loops'.

### 13.0 4Dii: Object-Wave Duality Surfaces

### 13.1 Observable and Embedded Dimensions

As stated in section
1.1 General Characteristics, a dimension is defined essentially as a variable and functions are categorized by the number of dimensions/variables that they use. Further, the functions are categorized by the number of real, versus imaginary, variables that they are using. And so, the notations, 3Di, 4Dii, 5Dii, are useful designations of groups of functions.

In our human perceptions we also have an experience of dimension that is more like direction and uses the words, horizontal, vertical, depth etc. The rationale behind equating a variable with a dimension comes into being at the time when there is an intention to make measurements along those or other 'directions'.

What we have been using so far in the graphs are:
i. the horizontal variable
ii. the vertical variable
iii. the depth variable
iv. the animation variable
v. the parametric variable(s)

Additionally, sometimes the parametric variables are graphed, and sometimes not.

I've, personally found useful the realization that there are essentially only four graphed, or observable, dimensions (first four above); and theoretically there could be many ungraphed, unobserved or embedded dimensions. This terminology provides additional information as to what a particular 'parameter' is doing, and so that is the rationale for using it.

In this section functions of the form:

$$
y+i z=f(u+i v)
$$

are explored with $u$ and $v$ alternating between the observable and the embedded roles. What is meant by object-wave duality is that when $u$ is observable and $v$ is embedded, a surface 'object' appears; and when $v$ is observable and $u$ is embedded, a surface 'wave' (helicoid) appears. Sometimes their roles are reversed. Also, in some cases, and in later sections, neither are playing an observable role, meaning that the observable dimensions are all functions of embedded ones.

### 13.2 Objects and Their Associated Waves

### 13.21 The Sphere Surface

From section 12.21 The Sphere Surface Function:

$$
y+i z=e^{i v}\left(1-u^{2}\right)^{\frac{1}{2}}
$$


uses $u$ on the horizontal or $x$ axis. Switching the horizontal axis to $x=v$, and using the same intervals $-1 \leq u \leq 1$ and $0 \leq v \leq 2 \pi$, results in a Helicoid wave.


The $v$ interval determines the overall length of the wave.

Extending it to $-3 \pi \leq v \leq 3 \pi$ :


And, the $u$ interval determines the amplitude. With these types of waves there is what we could call an 'inner amplitude', which extends from the helix amplitude at 1 , toward the $x$-axis and an 'outer amplitude', which extends from the helix amplitude at 1 , outward. The negative $u$ interval determines the inner amplitude surface, and the positive $u$ interval determines the outer amplitude surface. See section 13.31 Inner, Outer and Variable Amplitude

For example, with $-.65 \leq u \leq-.35$ :


### 13.22 The Reciprocal Sphere Surface

With the same intervals as for the sphere: $-1 \leq u \leq 1$ and $0 \leq v \leq 2 \pi$, the reciprocal of the sphere surface,

$$
\begin{gathered}
x=u \\
y+i z=\left(e^{i v}\left(1-u^{2}\right)^{\frac{1}{2}}\right)^{-1}
\end{gathered}
$$

is similar to a catenoid except that it is asymptotic and, therefore, not continuous at $\pm 1$.


Switching the horizontal axis to $x=v$, the reciprocal sphere generates its own wave.


As before, $v$ determines overall length; and $u$, staying inside the discontinuities, determines the amplitude, in this case, with the outer limit, determining outer amplitude that can increase without bound, and the lower limit determining inner amplitude around the circular opening.

With $.95 \leq u \leq .99$ and $-2 \pi \leq v \leq 2 \pi$ :


### 13.23 The Exp/Natural Log Surface

As in section 11.23 The Four Coordinate Complex Exp/Log Surface:

$$
y+i z=e^{u+i v}
$$

With $x=u$ and $-6 \leq u \leq 1.4,0 \leq v \leq 2 \pi$ :


When $x$ is switched to $v$ with these same intervals:


As before, $v$ determines the overall length. Extending $v$ to $-3 \pi \leq v \leq 3 \pi$ :


As before, the amplitude, determined by $u$, has an outer amplitude determined exponentially by positive $u$, and an inner amplitude that approaches zero but does not reach it, determined by negative $u$.

In the case where $u$ has no interval and is equal to zero, the helicoid becomes a helix. So, the two amplitudes are:

$$
\begin{gathered}
1 \leq \text { outer amplitude } \leq e^{u} \\
e^{-u}<\text { inner amplitude } \leq 1
\end{gathered}
$$

E.g., $0 \leq u \leq 1.4$ :


And, $-2 \leq u \leq 0$ :


So this type of helicoid always has a hole in it around the $x$ axis.

### 13.24 Polynomial Surface

From section 12.31 Constant, Exponential and Polynomial Surfaces, a cubic polynomial with a surface:

$$
y+i z=e^{i v}\left(u^{3}+u^{2}-u-1\right)
$$

in object mode with intervals $-2 \leq u \leq 2$ and $0 \leq v \leq 2 \pi$ :


In wave mode, $x=v$, results in a kind of double wave with the higher amplitude determined by the $+u$ part of the interval, and the lower amplitude determined by the $-u$ lower part of the interval. The $v$ interval is extended to $\pm 3 \pi$ :


### 13.25 Cardioid Surface

In the case of an $i$ base surface function, $v$ generates the object mode and $u$ the wave mode.

$$
\begin{gathered}
y+i z=i^{i v+i^{u-1}} \\
x=v \text { and } 0 \leq u \leq 4,0 \leq v \leq 4
\end{gathered}
$$



$$
\text { And, } x=u,-4 \leq u \leq 4
$$



### 13.3 Helicoids

### 13.31 Inner, Outer and Variable Amplitude

Looking more closely at the 'helicoid with hole', in wave mode, i.e., $x=v$, if $u=0$, then the surface helicoid is equivalent to the basic Euler helix. Meaning that:

$$
\begin{aligned}
x & =v \\
y+i z & =e^{u+i v} \\
u & =0
\end{aligned}
$$

is equivalent to:

$$
y+i z=e^{i x}
$$



If we then add a small positive $u$ interval, $0 \leq u \leq .5$, the outer amplitude shows up:


Or, with a small negative $u$ interval, $-.5 \leq u \leq 0$, the inner amplitude shows up:


The frequency is altered with a coefficient, $f$, on the exponent:

$$
\begin{gathered}
x=v \\
y+i z=e^{f(u+i v)} \\
f=2
\end{gathered}
$$



A variable amplitude wave results by making the base complex with the addition of an imaginary constant.

$$
\begin{gathered}
y+i z=(e+b i)^{f(u+i v)} \\
b=-.2 \\
-2 \pi \leq v \leq 2 \pi
\end{gathered}
$$



The above helicoid variable amplitude, with the function in 'wave mode', is reflected below in a 'nested variable amplitude' when the function is in 'object mode'.

Adjusting, $b=.4,-3 \leq u \leq 0,-5 \leq v \leq 4$, for a better view:


### 13.32 Geometric Helicoids

In wave mode, returning the frequency to 1 , adding the reciprocal sum second term, and adjusting amplitude coefficients and $v$ interval:

$$
\begin{gathered}
y+i z=A(e+b i)^{(u+i v)}+B(e+b i)^{-(u+i v)} \\
A=3, B=-1, b=-3,-10 \leq v \leq 9
\end{gathered}
$$

results in an elliptic helicoid:


Returning the base to real only, with $b=0$; adding frequency coefficients, $f=2, g=1$; and adjusting amplitude coefficients $A$ and $B$, results in a 'helitricoid'.

$$
\begin{gathered}
y+i z=A(e+b i)^{f(u+i v)}+B(e+b i)^{-g(u+i v)} \\
A=1, B=3.3, f=2, g=1, b=0, \quad-.5 \leq u \leq 0,-10 \leq v \leq 9
\end{gathered}
$$



The geometric shape can be placed either on the inner amplitude or the outer amplitude. With $f=3, g=1$, and $A=2.35, B=2.75$, the square is on the outer amplitude:


And, with $A=.35$, the square is on the inner amplitude:


### 13.4 A 'Transverse Wave' Surface

As might be imagined, venturing into the area of integral operations as surfacing operators brings forth an array of interesting puzzles requiring much analysis. So, here are just two wave forms using an integral operation as a surface operator on the basic half-circle function.

$$
y+i z=\left(1-u^{2}\right)^{\frac{1}{2}} \int_{0}^{v} e^{i v} d v
$$

Intervals $0 \leq u \leq 1.5,-3 \pi \leq v \leq 3 \pi$ :


And next, changing only the lower limit of integration with the same intervals.

$$
y+i z=\left(1-u^{2}\right)^{\frac{1}{2}} \int_{-v}^{v} e^{i v} d v
$$



As usual, $u$ determines the amplitude, and seems to change direction from horizontal to vertical as $u$ transitions through 1 . Additionally, the transition seems to involve the inner amplitude on one side, and the outer amplitude on the other side.

### 14.0 5Dii: Circular Surface Functions

In 5Dii, the variable $x$ becomes a parametric 'output only' giving three output, or observable, dimensions along with two input, or embedded, dimensions. $x$, therefore, becomes a function of ( $u+i v$ ) along with $y$ and $i z$, generating a parametric system:

$$
\text { 5Dii } \quad \begin{aligned}
& x=f(u+i v) \\
& y=g(u+i v) \\
& i z=h(u+i v)
\end{aligned}
$$

### 14.1 Circular Helix

The basic Euler Helix with frequency $f$ :

$$
\begin{gathered}
y+i z=e^{f i x} \\
f=16
\end{gathered}
$$


can be made circular by arranging three functions: a geometry function, an orbiting function, and a 'rotational function'; and then, taking the various real and imaginary parts of each in different ways, to give the three output axes.

In this case, the geometry function is a simple circle with a frequency:

$$
G=e^{16 i t}
$$

The orbiting function is also a simple circle:

$$
O=b e^{i t}
$$

$b=$ radius of the orbit.

If these two are graphed in the following way:

$$
\begin{gathered}
\hline x=\text { real } O+\operatorname{imag} G \\
y=\text { realG } \\
i z=\operatorname{imag} O+\operatorname{imag} G \\
\hline
\end{gathered}
$$

the helix goes in a circle, but the movement of the two circular functions, geometry and orbit, are not properly coordinated resulting in:


So, a rotator function is needed to coordinate the two movements:

$$
R=e^{i t}
$$

And, this is applied to the imaginary component of the geometry function.

$$
\begin{array}{|c|}
\hline x=\text { real } O+R \operatorname{imag} G \\
y=\text { realG } \\
i z=\operatorname{imag} O+R \operatorname{imag} G \\
\hline
\end{array}
$$

And then, the real and imaginary components of this are applied to the $x$ and iz axes respectively. The vertical axis, $y$, is not effected by adding an orbit to the helix:

$$
\begin{gathered}
x=\text { real } O+\operatorname{real}(R \operatorname{imag} G) \\
y=\operatorname{real} G \\
i z=\operatorname{imag} O+\operatorname{imag}(R \operatorname{imag} G)
\end{gathered}
$$

And, the circular helix results:


With various values of the radius, $b$, in the orbit function, the usual torus shapes are formed:
the sphere with $b=0$, the spindle torus with $b=.8$, and the circular helix with $b=4$ :


Animation 72 'Circular Helix Torus Variations'

As with the other helix functions, various shapes can be applied to the Geometry and Orbiting functions using reciprocal sums, frequency coefficients, and amplitude adjustments.


Animation 73 'Circular Helix Shapes'

### 14.2 Circular Helicoid

To place a surface on the circular helix and generate a circular helicoid, all that's needed is to convert the three functions:

$$
\begin{gathered}
x=\operatorname{realO}+\operatorname{real}(R \operatorname{imag} G) \\
y=\operatorname{real} G \\
i z=\operatorname{imag} O+\operatorname{imag}(R \operatorname{imag} G)
\end{gathered}
$$

$$
\begin{gathered}
G=e^{f i t} \\
O=b e^{i t} \\
b=\text { radius of the orbit. }
\end{gathered}
$$

$$
R=e^{i t}
$$

to complex regional domain input, and add a $u$ interval - in this case, an inner amplitude:

$$
\begin{gathered}
G=e^{f(u+i v)} \\
O=b e^{i v} \\
R=e^{i v} \\
f=6, \quad b=4, \quad-4 \leq u \leq 0, \quad 0 \leq v \leq 2 \pi
\end{gathered}
$$

The helix outlines the helicoid in red:


### 14.3 Geometric Torus Surfaces

The geometric torus surfaces functions on the cover are built in the same basic way as the helicoid with variations, using the same parametric functions. For example, a triangular geometry in a square orbit, with the geometry function in $e$ base, and the orbit and rotator functions in $i$ base:

$$
\begin{gathered}
x=\text { real } O+\text { real }(R \operatorname{imag} G) \\
y=\text { real } G \\
i z=\operatorname{imag} O+\operatorname{imag}(R \operatorname{imag} G)
\end{gathered}
$$

$$
\begin{gathered}
G=A e^{a i v}+B e^{b i v} \\
O=C i^{c u}+D i^{d u} \\
R=i^{u} \\
A=.325, B=1.35, C=-7.12, D=.975 \\
0 \leq u \leq 3.5,0 \leq v \leq 2 \pi
\end{gathered}
$$

The $u$ interval is 3.5 for cutaway, and 4.0 for complete:


By adjusting coefficients, any number of cusps, loops, Trochoid and Cycloid shapes are possible.
with: $A=1.35$ :


With $A=.85, B=3, D=0, a=7$ :


### 14.4 Circular 'Transverse Wave' Surfaces

Here is the same basic principle applied to the two waveforms from section 13.4 A 'Transverse Wave' Surface:

$$
\begin{gathered}
x=\text { real } O+\text { real }(R \operatorname{imag} G) \\
y=\text { real } G \\
i z=\operatorname{imag} O+\operatorname{imag}(R \operatorname{imag} G)
\end{gathered}
$$

$$
\begin{gathered}
G=\left(1-u^{2}\right)^{\frac{1}{2}} \int_{0}^{v} e^{f i v} d v \\
O=b e^{i v} \\
R=e^{i v} \\
f=4, b=1 \\
0 \leq u \leq 1.1, \quad 0 \leq v \leq 2 \pi
\end{gathered}
$$



And:

$$
G=\left(1-u^{2}\right)^{\frac{1}{2}} \int_{-v}^{v} e^{f i v} d v
$$



### 15.0 6Dii: Surfaces in Motion

Surface functions in 6Dii are of the form:

$$
y+z i=f(u+v i, t)
$$

And, $x$ is generally 'output only' and specified parametrically.

$$
\begin{array}{|l|}
\hline x=f(u+i v, t) \\
y=g(u+i v, t) \\
i z=h(u+i v, t)
\end{array}
$$

We now have enough variables to generate our own closed surfaces and put them in motion. The parametric functions specifying the observable dimensions will include geometry functions, surfacing functions, and trajectory functions.

### 15.1 Objects in Polynomial Space Trajectories

## In section

5.1 An Intuitive Model

## Complex Slope

Using the animation above, visualize an aircraft taxiing down the runway prior to take-off. Our view is off to the side, with the taxiing aircraft moving from left to right. And, let's say that exactly to the right is a heading of zero. Exactly in front of us, the aircraft reaches take-off speed and rotates to begin its climb. This is the violet ball at the origin. The violet line is the aircraft's climb while maintaining the same heading. This is real slope and zero imaginary slope, sometimes referred to as 'rise over run.'

Next, at the black ball, the aircraft reaches cruising altitude and levels off while maintaining the same heading. And, the black line shows its flight path with zero real slope and zero imaginary slope.

Next, at the red ball, the aircraft executes a 45 -degree turn to the left while maintaining altitude. The red line shows its flight path with imaginary slope and zero real slope. This could be referred to as 'glide over run.'

And finally, while maintaining that heading, at the blue ball, it begins another climb. The blue line then shows both real slope, which is the climb, and imaginary slope, which is the heading other than zero. So, in flight path terms, complex slope is the sum of climb/descent plus heading.

### 5.2 Real, Imaginary and Complex Slope

Removing the idea of an aircraft, since it has a direction and motion, and just focusing on the line segments, real only slope of a line in three dimensions is:
while imaginary only slope is:


Real only slope appears in the front view (FRP) and imaginary only slope appears in the top view (TIP).

Real only slope has the usual slope-intercept equation of a line:

$$
y=m x+b_{y} \quad b_{y}=y \text { intercept }
$$

and imaginary only slope would then have a corresponding slope-intercept equation of a line:

$$
i z=i m x+i b_{i z} \quad i b_{i z}=i z \text { intercept }
$$

In graphing terms, real only slope is rise over run and imaginary slope would be glide over run. Either or both can be positive, negative or zero. Complex slope combines the two and is 'rise plus glide over run'. The two equations can be combined to give:

$$
y+i z=\left(m_{r}+i m_{i}\right) x+b_{y}+i b_{i z}
$$

Algebraically, complex slope extends standard slope by adding in the imaginary number for the glide. Since there are two slopes:
$m_{r}=$ the real component of complex slope, the rise in FRP im $_{i}=$ the imaginary component of complex slope, the glide in TIP

And the calculation of complex slope becomes:

$$
m_{r}+i m_{i}=\frac{\left(y_{2}+i z_{2}\right)-\left(y_{1}+i z_{1}\right)}{x_{2}-x_{1}}
$$

Also, rather than an axis intercept, there is a displacement of the line relative to both the $y$-axis and the $i z$-axis. Meaning there is a real displacement and an imaginary displacement. So the complete equation is:

$$
\begin{gathered}
y+i z=\left(m_{r}+i m_{i}\right) x+d_{r}+i d_{i} \\
d_{r}+i d_{i}=\text { complex displacement of the line }
\end{gathered}
$$

The real displacement moves the line up and down. The imaginary displacement moves the line forward and backward.

### 5.21 Example:

What is the equation of the line that goes through the two points: $(3,2, i)$ and $(1,-3,6 i)$ ?
The first step is to calculate the two slopes, real and imaginary:

$$
\begin{gathered}
m_{r}+i m_{i}=\frac{\left(y_{2}+i z_{2}\right)-\left(y_{1}+i z_{1}\right)}{x_{2}-x_{1}} \\
m_{r}+i m_{i}=\frac{(-3+6 i)-(2+i)}{1-3} \\
m_{r}+i m_{i}=\frac{-5+5 i}{-2} \\
m_{r}=\frac{5}{2} \quad i m_{i}=\frac{-5 i}{2}
\end{gathered}
$$

(The two slopes, of course, need not be equal. This example just turned out that way.)

The second step is to insert the slopes along with either point into the basic equation to solve for the displacements. Using the first point:

$$
\begin{gathered}
y+i z=\left(m_{r}+i m_{i}\right) x+d_{r}+i d_{i} \\
2+i=(2.5-2.5 i) 3+d_{r}+i d_{i} \\
2+i=(7.5-7.5 i)+d_{r}+i d_{i} \\
-5.5+8.5 i=d_{r}+i d_{i r}
\end{gathered}
$$

The third step, if needed, is to insert the slopes and the second point into the basic equation to verify that the two points give the same displacements.

$$
\begin{gathered}
y+i z=\left(m_{r}+i m_{i}\right) x+d_{r}+i d_{i} \\
-3+6 i=(2.5-2.5 i)+d_{r}+i d_{i} \\
-5.5+8.5 i=d_{r}+i d_{i}
\end{gathered}
$$

And so, the completed equation for the line with the two specified points is:

$$
\begin{gathered}
y+i z=\left(m_{r}+i m_{i}\right) x+d_{y}+i d_{i z} \\
y+i z=(2.5-2.5 i) x-5.5+8.5 i
\end{gathered}
$$

When this line is projected to the front view (real only slope) it appears as:


When the line is projected to the top view (imaginary only slope) it appears as:


The 3Di graph of the line along with the two specified points is as follows:

rotate complex slope line

### 5.3 Inverse Imaginary Slope

In addition to real slope in the FRP and imaginary slope in the TIP, denoted by:

$$
\begin{aligned}
& m_{r}+i m_{i}=\frac{\left(y_{2}+i z_{2}\right)-\left(y_{1}+i z_{1}\right)}{x_{2}-x_{1}} \\
& m_{r}=\frac{\left(y_{2}-y_{1}\right)}{x_{2}-x_{1}} \quad i m_{i}=\frac{\left(i z_{2}-i z_{1}\right)}{x_{2}-x_{1}}
\end{aligned}
$$

we can define an 'inverse imaginary slope':

$$
i_{i i}=\text { inverse imaginary slope }
$$

and denote it by:

$$
i m_{i i}=\frac{y_{2}-y_{1}}{i z_{2}-i z_{1}}
$$

Real slope shows up in the front view and imaginary slope shows up in the top view. Inverse imaginary slope shows up in the right side view.

One way that this can be visualized is by standing at the end of a runway while the aircraft takes off going away from us. In this front view the aircraft appears to rise vertically. This vertical ascent appearance occurs in both front and top views. And, this demonstrates that a line with inverse imaginary slope, so defined, appears as a vertical line in both the FRP and the TIP. Intuitively, we can carry these visualizations further to formally observe that:
(4) A line with real only slope shows up as a vertical line in the side view, and a line with zero slope in the top view.
(5) A line with imaginary only slope shows up as a line with zero slope in front view, and a line with zero slope in side view.
(6) And, as stated above, a line with inverse imaginary only slope shows up as a vertical line in both front and top views.

Inverse imaginary slope may appear somewhat counter intuitive in that the glide path of the above mentioned aircraft would have a positive inverse imaginary slope on landing/approach, and a negative inverse imaginary slope on take-off/departure.

Continuing the example with the two previously specified points, $(3,2, i)$ and $(1,-3,6 i)$, the inverse imaginary slope can be calculated as:

$$
\begin{gathered}
i m_{i i}=\frac{y_{2}-y_{1}}{i z_{2}-i z_{1}} \\
i m_{i i}=\frac{-3-2}{6 i-i}=\frac{-5}{5 i} \\
i m_{i i}=\frac{-1}{i} \quad \text { or } \quad i m_{i i}=i
\end{gathered}
$$

This can be viewed, in the above last animation, as the blue line comes around to show the RSIP view; and it can be verified by projecting the line (blue) to the RSIP as follows. In the side view the axes are:

$$
(\text { horizontal, vertical })=(i z, y)
$$

and the two displacements, when combined, project to a $y$-intercept that is different. By inserting the two points into the equation:

$$
y=i m_{i i}(i z)+b_{y}
$$

the $y$-intercept is calculated as:

$$
\begin{gathered}
\text { 1st point: }(i z, y)=(i, 2) \\
2=i(i)+b_{y} \quad 3=b_{y} \\
2 n d \text { point: }(i z, y)=(6 i,-3) \\
-3=i(6 i)+b_{y} \quad 3=b_{y}
\end{gathered}
$$

So the equation of this line is:

$$
y=i(i z)+3
$$

And when this line is projected to the right side view it appears as:


The three different two dimensional graphs are generated by the following equations:

> Front view $\quad y=m_{r} x+d_{r}$
> Top view $\quad i z=i m_{i} x+i d_{i}$
> Right side view $\quad y=i m_{i i} i z+b_{y}$

### 5.4 Table of Slopes in 3Di

Additionally, the three 2-dimensional slopes can also be visualized as the three planar rotations of a spacecraft. I.e., pitch, yaw and roll which is indicated in the fourth column of the table:

| slope | notation | plane | slope <br> rotation | action | 2D relationships |
| :--- | :---: | :--- | :--- | :--- | :--- |
| complex | $m_{r}+i m_{i}$ |  | pitch + <br> yaw | rise + glide <br> over run | Slope in all <br> three, FRP, TIP, <br> RSIP |
| real only | $m_{r}$ | FRP | pitch | rise over run | Horizontal in TIP, <br> vertical in RSIP |
| imaginary <br> only | $i m_{i}$ | TIP | yaw | glide over run | Horizontal in <br> both FRP and <br> RSIP |
| inverse <br> imaginary | $i m_{i i}$ | RSIP | roll | rise over glide | Vertical line in <br> both FRP and TIP |

### 5.5 Transformation of Two Dimensional Slope

## If we specify a point at $(1,1 / 2,0 i)$ :


and then draw a line through this point with no displacement, meaning a line through this point and the origin:


The equation of this line is generated by:

$$
y+i z=\left(m_{r}+i m_{i}\right) x+d_{r}+i d_{i}
$$

With no displacement, zero imaginary slope and an arbitrary $1 / 2$ real slope, the equation reduces to:

$$
y=\frac{x}{2}
$$

which is a line with real-only slope located on the front, real plane (FRP).

If we were to rotate this line about the $x$-axis using a 'rotator coefficient' $i^{a}$ (see sections: 3.25, 3.7, 9.38 on this rotator in the eBook), the effect is to alter the line's two dimensional real and imaginary slopes. Meaning:

$$
\begin{gathered}
m_{r}+i m_{i}=i^{a} \\
0 \leq a \leq 4 \\
y+i z=\frac{\left(m_{r}+i m_{i}\right) x}{2}=\frac{i^{a} x}{2}
\end{gathered}
$$

So as a moves through its interval, the line is rotated about the $x$-axis:

line rotation about the $x$-axis

As the line rotates, its projected two dimensional slope transforms from real-only to complex to imaginary only to complex. And then to negative real-only to complex to negative imaginaryonly to complex and then back again to positive real-only. So if we look at various values of $a$ : For $a=0$ the slope is positive real only:

$$
y=\frac{x}{2}
$$

On the right, the black line is the two dimensional real slope in the front view and the red line is the two dimensional imaginary slope in the top view:



For $a=.6$ the slope is complex:

$$
y+i z=\frac{(.58779+.80902 i)}{2} x
$$



For $a=1$ the slope is positive imaginary only:

$$
i z=\frac{i}{2} x
$$



For $a=1.6$ the slope is complex with negative real and positive imaginary:

$$
y+i z=\frac{(-.80902+.29390 i)}{2} x
$$



For $a=2.6$ the slope is complex with negative real and negative imaginary:

$$
y+i z=\frac{(-.58779-.80902 i)}{2} x
$$



For $a=3$ the slope is negative imaginary only:

$$
i z=\frac{-i}{2} x
$$



As stated before, if a displacement is added to the line, a real displacement moves the line up and down and an imaginary displacement moves the line forward or backward. Then the line will rotate at the displacement point about a line (green line below) through that point and parallel to the x -axis. For example, adding a displacement and using the above line with $a=3$ :

$$
y+i z=\frac{-i}{2} x+1+i
$$



Just as this line, which is rotated about the x-axis, transitions between real and imaginary slope, if the line is rotated about the iz-axis the slope will transition between real and inverse imaginary slopes. Similarly if the line is rotated about the $y$-axis the line will transition between imaginary and inverse imaginary slopes.
5.6 Polynomial Space Trajectories, we made the suggestion that polynomials in space with complex coefficients can be interpreted as three-dimensional trajectories along which objects can travel - meaning that, essentially, complex coefficients are slopes.

A simple line with complex slope and a complex displacement:

$$
y+i z=\left(m_{r}+i m_{i}\right) x+d_{y}+i d_{i z}
$$

with the same two points from section 5.0 Complex Slope, gives the equation:

$$
y+i z=(2.5-2.5 i) x-5.5+8.5 i
$$

which generates the line in space and the two points:


And, adding an $x^{2}$ term:

$$
y+i z=x^{2}+(2.5-2.5 i) x-5.5+8.5 i
$$

generates a space parabola:


Virtually any combination of slopes (complex coefficients) and polynomial terms can be combined to form a trajectory. Placing this particular slope on a cubic term, and adding an arbitrary slope on a linear term with no displacement, for example, gives the space curve below in blue:

$$
y+i z=(2.5-2.5 i) x^{3}-(2-6 i) x
$$



Or, adding back in the displacement, and raising the degree:

$$
y+i z=(2.5-2.5 i) x^{4}-(2-6 i) x^{2}-5.5+8.5 i
$$


we then designate this as the trajectory, $T$ :
greg ehmka, 2013

$$
T=(2.5-2.5 i) x^{4}-(2-6 i) x^{2}-5.5+8.5 i
$$

And then, the spatial dimension $x$, which gives the line graph, needs to be replaced by an animation dimension; and, in this case, we'll say that it is time, $t$ :

$$
T=(2.5-2.5 i) t^{4}-(2-6 i) t^{2}-5.5+8.5 i
$$

Rather than representing the spatial path with $x$, using an animation variable, $t$ allows the trajectory function to represent the object's motion, i.e., the various velocities and accelerations which will move an object along that spatial path.

The replacement of $x$ by $t$ doesn't change the essential nature of the equation. That being, a location graph. Using the spatial coordinate $x$ gives a line graph representing all possible locations. Using the time coordinate $t$ gives a video of successive unique locations at time $=t$.

The next step is to provide an object that will travel this path. As we indicated in section 5.1 An Intuitive Model

## Complex Slope

Using the animation above, visualize an aircraft taxiing down the runway prior to take-off. Our view is off to the side, with the taxiing aircraft moving from left to right. And, let's say that exactly to the right is a heading of zero. Exactly in front of us, the aircraft reaches take-off speed and rotates to begin its climb. This is the violet ball at the origin. The violet line is the aircraft's climb while maintaining the same heading. This is real slope and zero imaginary slope, sometimes referred to as 'rise over run.'

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And finally, while maintaining that heading, at the blue ball, it begins another climb. The blue line then shows both real slope, which is the climb, and imaginary slope, which is the heading other than zero. So, in flight path terms, complex slope is the sum of climb/descent plus heading.

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while imaginary only slope is:


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$$
y=m x+b_{y} \quad b_{y}=y \text { intercept }
$$

and imaginary only slope would then have a corresponding slope-intercept equation of a line:

$$
i z=i m x+i b_{i z} \quad i b_{i z}=i z \text { intercept }
$$

In graphing terms, real only slope is rise over run and imaginary slope would be glide over run. Either or both can be positive, negative or zero. Complex slope combines the two and is 'rise plus glide over run'. The two equations can be combined to give:

$$
y+i z=\left(m_{r}+i m_{i}\right) x+b_{y}+i b_{i z}
$$

Algebraically, complex slope extends standard slope by adding in the imaginary number for the glide. Since there are two slopes:
$m_{r}=$ the real component of complex slope, the rise in FRP im $_{i}=$ the imaginary component of complex slope, the glide in TIP

And the calculation of complex slope becomes:

$$
m_{r}+i m_{i}=\frac{\left(y_{2}+i z_{2}\right)-\left(y_{1}+i z_{1}\right)}{x_{2}-x_{1}}
$$

Also, rather than an axis intercept, there is a displacement of the line relative to both the $y$-axis and the $i z$-axis. Meaning there is a real displacement and an imaginary displacement. So the complete equation is:

$$
\begin{gathered}
y+i z=\left(m_{r}+i m_{i}\right) x+d_{r}+i d_{i} \\
d_{r}+i d_{i}=\text { complex displacement of the line }
\end{gathered}
$$

The real displacement moves the line up and down. The imaginary displacement moves the line forward and backward.

### 5.21 Example:

What is the equation of the line that goes through the two points: $(3,2, i)$ and $(1,-3,6 i)$ ?
The first step is to calculate the two slopes, real and imaginary:

$$
\begin{gathered}
m_{r}+i m_{i}=\frac{\left(y_{2}+i z_{2}\right)-\left(y_{1}+i z_{1}\right)}{x_{2}-x_{1}} \\
m_{r}+i m_{i}=\frac{(-3+6 i)-(2+i)}{1-3} \\
m_{r}+i m_{i}=\frac{-5+5 i}{-2} \\
m_{r}=\frac{5}{2} \quad i m_{i}=\frac{-5 i}{2}
\end{gathered}
$$

(The two slopes, of course, need not be equal. This example just turned out that way.)

The second step is to insert the slopes along with either point into the basic equation to solve for the displacements. Using the first point:

$$
\begin{gathered}
y+i z=\left(m_{r}+i m_{i}\right) x+d_{r}+i d_{i} \\
2+i=(2.5-2.5 i) 3+d_{r}+i d_{i} \\
2+i=(7.5-7.5 i)+d_{r}+i d_{i} \\
-5.5+8.5 i=d_{r}+i d_{i r}
\end{gathered}
$$

The third step, if needed, is to insert the slopes and the second point into the basic equation to verify that the two points give the same displacements.

$$
\begin{gathered}
y+i z=\left(m_{r}+i m_{i}\right) x+d_{r}+i d_{i} \\
-3+6 i=(2.5-2.5 i)+d_{r}+i d_{i} \\
-5.5+8.5 i=d_{r}+i d_{i}
\end{gathered}
$$

And so, the completed equation for the line with the two specified points is:

$$
\begin{gathered}
y+i z=\left(m_{r}+i m_{i}\right) x+d_{y}+i d_{i z} \\
y+i z=(2.5-2.5 i) x-5.5+8.5 i
\end{gathered}
$$

When this line is projected to the front view (real only slope) it appears as:


When the line is projected to the top view (imaginary only slope) it appears as:


The 3Di graph of the line along with the two specified points is as follows:

rotate complex slope line

### 5.3 Inverse Imaginary Slope

In addition to real slope in the FRP and imaginary slope in the TIP, denoted by:

$$
\begin{aligned}
& m_{r}+i m_{i}=\frac{\left(y_{2}+i z_{2}\right)-\left(y_{1}+i z_{1}\right)}{x_{2}-x_{1}} \\
& m_{r}=\frac{\left(y_{2}-y_{1}\right)}{x_{2}-x_{1}} \quad i m_{i}=\frac{\left(i z_{2}-i z_{1}\right)}{x_{2}-x_{1}}
\end{aligned}
$$

we can define an 'inverse imaginary slope':

$$
i_{i i}=\text { inverse imaginary slope }
$$

and denote it by:

$$
i m_{i i}=\frac{y_{2}-y_{1}}{i z_{2}-i z_{1}}
$$

Real slope shows up in the front view and imaginary slope shows up in the top view. Inverse imaginary slope shows up in the right side view.

One way that this can be visualized is by standing at the end of a runway while the aircraft takes off going away from us. In this front view the aircraft appears to rise vertically. This vertical ascent appearance occurs in both front and top views. And, this demonstrates that a line with inverse imaginary slope, so defined, appears as a vertical line in both the FRP and the TIP. Intuitively, we can carry these visualizations further to formally observe that:
(7) A line with real only slope shows up as a vertical line in the side view, and a line with zero slope in the top view.
(8) A line with imaginary only slope shows up as a line with zero slope in front view, and a line with zero slope in side view.
(9) And, as stated above, a line with inverse imaginary only slope shows up as a vertical line in both front and top views.

Inverse imaginary slope may appear somewhat counter intuitive in that the glide path of the above mentioned aircraft would have a positive inverse imaginary slope on landing/approach, and a negative inverse imaginary slope on take-off/departure.

Continuing the example with the two previously specified points, $(3,2, i)$ and $(1,-3,6 i)$, the inverse imaginary slope can be calculated as:

$$
\begin{gathered}
i m_{i i}=\frac{y_{2}-y_{1}}{i z_{2}-i z_{1}} \\
i m_{i i}=\frac{-3-2}{6 i-i}=\frac{-5}{5 i} \\
i m_{i i}=\frac{-1}{i} \quad \text { or } \quad i m_{i i}=i
\end{gathered}
$$

This can be viewed, in the above last animation, as the blue line comes around to show the RSIP view; and it can be verified by projecting the line (blue) to the RSIP as follows. In the side view the axes are:

$$
(\text { horizontal, vertical })=(i z, y)
$$

and the two displacements, when combined, project to a y-intercept that is different. By inserting the two points into the equation:

$$
y=i m_{i i}(i z)+b_{y}
$$

the $y$-intercept is calculated as:

$$
\begin{gathered}
\text { 1st point: }(i z, y)=(i, 2) \\
2=i(i)+b_{y} \quad 3=b_{y} \\
2 n d \text { point: }(i z, y)=(6 i,-3) \\
-3=i(6 i)+b_{y} \quad 3=b_{y}
\end{gathered}
$$

So the equation of this line is:

$$
y=i(i z)+3
$$

And when this line is projected to the right side view it appears as:


The three different two dimensional graphs are generated by the following equations:

> Front view $\quad y=m_{r} x+d_{r}$
> Top view $\quad i z=i m_{i} x+i d_{i}$
> Right side view $\quad y=i m_{i i} i z+b_{y}$

### 5.4 Table of Slopes in 3Di

Additionally, the three 2-dimensional slopes can also be visualized as the three planar rotations of a spacecraft. I.e., pitch, yaw and roll which is indicated in the fourth column of the table:

| slope | notation | plane | slope <br> rotation | action | 2D relationships |
| :--- | :---: | :--- | :--- | :--- | :--- |
| complex | $m_{r}+i m_{i}$ |  | pitch + <br> yaw | rise + glide <br> over run | Slope in all <br> three, FRP, TIP, <br> RSIP |
| real only | $m_{r}$ | FRP | pitch | rise over run | Horizontal in TIP, <br> vertical in RSIP |
| imaginary <br> only | $i m_{i}$ | TIP | yaw | glide over run | Horizontal in <br> both FRP and <br> RSIP |
| inverse <br> imaginary | $i m_{i i}$ | RSIP | roll | rise over glide | Vertical line in <br> both FRP and TIP |

### 5.5 Transformation of Two Dimensional Slope

## If we specify a point at $(1,1 / 2,0 i)$ :


and then draw a line through this point with no displacement, meaning a line through this point and the origin:


The equation of this line is generated by:

$$
y+i z=\left(m_{r}+i m_{i}\right) x+d_{r}+i d_{i}
$$

With no displacement, zero imaginary slope and an arbitrary $1 / 2$ real slope, the equation reduces to:

$$
y=\frac{x}{2}
$$

which is a line with real-only slope located on the front, real plane (FRP).

If we were to rotate this line about the $x$-axis using a 'rotator coefficient' $i^{a}$ (see sections: 3.25, 3.7, 9.38 on this rotator in the eBook), the effect is to alter the line's two dimensional real and imaginary slopes. Meaning:

$$
\begin{gathered}
m_{r}+i m_{i}=i^{a} \\
0 \leq a \leq 4 \\
y+i z=\frac{\left(m_{r}+i m_{i}\right) x}{2}=\frac{i^{a} x}{2}
\end{gathered}
$$

So as a moves through its interval, the line is rotated about the $x$-axis:

line rotation about the $x$-axis

As the line rotates, its projected two dimensional slope transforms from real-only to complex to imaginary only to complex. And then to negative real-only to complex to negative imaginaryonly to complex and then back again to positive real-only. So if we look at various values of $a$ : For $a=0$ the slope is positive real only:

$$
y=\frac{x}{2}
$$

On the right, the black line is the two dimensional real slope in the front view and the red line is the two dimensional imaginary slope in the top view:


For $a=.6$ the slope is complex:

$$
y+i z=\frac{(.58779+.80902 i)}{2} x
$$



For $a=1$ the slope is positive imaginary only:

$$
i z=\frac{i}{2} x
$$



For $a=1.6$ the slope is complex with negative real and positive imaginary:

$$
y+i z=\frac{(-.80902+.29390 i)}{2} x
$$



For $a=2.6$ the slope is complex with negative real and negative imaginary:

$$
y+i z=\frac{(-.58779-.80902 i)}{2} x
$$



For $a=3$ the slope is negative imaginary only:

$$
i z=\frac{-i}{2} x
$$



As stated before, if a displacement is added to the line, a real displacement moves the line up and down and an imaginary displacement moves the line forward or backward. Then the line will rotate at the displacement point about a line (green line below) through that point and parallel to the $x$-axis. For example, adding a displacement and using the above line with $a=3$ :

$$
y+i z=\frac{-i}{2} x+1+i
$$



Just as this line, which is rotated about the x-axis, transitions between real and imaginary slope, if the line is rotated about the iz-axis the slope will transition between real and inverse imaginary slopes. Similarly if the line is rotated about the $y$-axis the line will transition between imaginary and inverse imaginary slopes.
5.6 Polynomial Space Trajectories, we can now use a closed surface object from section 12.2 Closed Surfaces. And, in this case, we'll take the triangular lozenge from section 12.22. So, the geometry function, $G$, will be:

$$
\begin{gathered}
G=\left(A e^{a i v}+B e^{b i v}\right)\left(1-u^{2}\right)^{\frac{1}{2}} \\
A=1.375, B=-.45, a=-1, b=2 \\
-1 \leq u \leq 1, \quad 0 \leq v \leq 2 \pi
\end{gathered}
$$

And, the closed surface graph of the object is:

$$
\begin{gathered}
x=u \\
y+i z=G
\end{gathered}
$$



The motion of this object along the trajectory $T$ is then just a simple sum of the geometry and the trajectory functions. Along the $x$ axis is the sum of the horizontal spatial extension of the object, plus the animation variable, $t$. And so:

$$
\begin{gathered}
x=u+t \\
y+i z=G+T
\end{gathered}
$$

The time interval is $-2 \leq t \leq 2$ :

## Animation 74 'Triangular Lozenge in Space Trajectory Travel'

And, overlaying the spatial path:


And here is an exotic trajectory with fractional exponents and a circular coefficient on the second term:

$$
T=(2.5-2.5 i) t^{3.6}-e^{4 i t}(2-6 i) t^{1.3}-5.5+8.5 i
$$



Animation 75 'Triangular Lozenge in Space Trajectory Travel 2'

### 15.2 Objects in Orbits

### 15.21 A Two Dimensional Solar System

For closed surfaces in orbit, first, each orbiting body will have its own geometry function. For example, with four bodies, a 'Sun,' a 'Planet,' a 'Moon,' and a 'Satellite' orbiting the 'Moon,' the four geometry functions will be, in this case, spheres of varying sizes:

$$
\begin{gathered}
G_{S}=e^{i v}\left(1-u^{2}\right)^{\frac{1}{2}} \\
G_{P}=e^{i v}\left(\left(\frac{1}{2}\right)^{2}-\left(\frac{u}{2}\right)^{2}\right)^{\frac{1}{2}} \\
G_{M}=e^{i v}\left(\left(\frac{1}{5}\right)^{2}-\left(\frac{u}{5}\right)^{2}\right)^{\frac{1}{2}}
\end{gathered}
$$

$$
\begin{aligned}
& G_{s}=e^{i v}\left(\left(\frac{1}{10}\right)^{2}-\left(\frac{u}{10}\right)^{2}\right)^{\frac{1}{2}} \\
& -1 \leq u \leq 1, \quad 0 \leq v \leq 2 \pi
\end{aligned}
$$

Next, the four trajectory functions. In this case the 'Sun' trajectory will be a reciprocal sum to allow for elliptical orbits, and the other three will be simple circular orbits. The animation variable will be time, $t$ :

$$
T_{S}=\left(a e^{i t}+(a+3) e^{-i t}\right)
$$

The coefficient $a$ allows for differing eccentricities in the elliptical orbit; and the constant 3 displaces the 'Sun' from the fixed center, the origin, around which it will be orbiting.

$$
\begin{aligned}
& T_{P}=b e^{f i t} \\
& T_{M}=c e^{g i t} \\
& T_{S}=d e^{h i t}
\end{aligned}
$$

The coefficients $b, c, d$ are the radii, or displacements, away from each preceding body; and the exponential coefficients $f, g, h$ are the respective frequencies of revolution around each preceding body.

The equations of motion are then the sums of each geometry function plus the orbiting trajectory functions from each preceding body plus adding its own orbiting trajectory to the preceding ones.

$$
\begin{gathered}
y+i z=G_{S}+T_{S} \text { for the 'Sun' } \\
y+i z=G_{P}+T_{S}+T_{P} \text { for the 'Planet' } \\
y+i z=G_{M}+T_{S}+T_{P}+T_{M} \text { for the 'Moon' } \\
y+i z=G_{S}+T_{S}+T_{P}+T_{M}+T_{S} \text { for the 'satellite' }
\end{gathered}
$$

For the displacements (radii) and frequencies:

$$
b=4, f=2, c=1.3, g=4, d=.4, h=5
$$

The different sizes of the spheres are graphed on the horizontal axis for motion in a vertical plane, and on the vertical axis for motion in the horizontal plane. In other words, the spatial extension of the sphere has two components - the real part of the geometry function and the imaginary part of the geometry function. These two parts are arranged on the three axes in such a way as to be both parallel and perpendicular to the plane of motion, so they move as desired.

As confusing as that may sound, it's fairly simple and, parametrically, looks like this:

In the case of the planet:

$$
\begin{gathered}
x=\operatorname{real}\left(G_{P}+T_{s}+T_{P}\right) \\
y=\frac{u}{5} \\
i z=\operatorname{imag}\left(G_{P}+T_{S}+T_{P}\right)
\end{gathered}
$$

And, there is one of these sets of equations for each moving body.

So, for motion in the horizontal plane, $x$ and $y$ are exchanging places, and so:

$$
\text { The vertical, } y=u, \frac{u}{2}, \frac{u}{5}, \frac{u}{10}
$$

The horizontal $x=$ real part of the motion equaion The depth iz = imagianry part of the motion equation

Interval $0 \leq t \leq 10$ :


Animation 76 '2D Solar System'

And, the spatial orbital paths look like this:


Or, with a different solar eccentricity and some frequency changes:

$$
a=-7, f=3, g=4, h=6
$$



Animation 77 '2D Solar System 2'

### 15.22 Bodily Rotation

A rotation can be added to each of the orbiting bodies by placing a 'spin coefficient', $e^{j i t}$, on the geometry function with, $j$, the frequency of revolution.

Adding rotation to the first three bodies:

$$
G_{S}=e^{j i t}\left[e^{i v}\left(1-u^{2}\right)^{\frac{1}{2}}\right] j=6
$$

$$
\begin{aligned}
& G_{P}=e^{j i t}\left[e^{i v}\left(\left(\frac{1}{2}\right)^{2}-\left(\frac{u}{2}\right)^{2}\right)^{\frac{1}{2}}\right] j=12 \\
& G_{M}=e^{j i t}\left[e^{i v}\left(\left(\frac{1}{5}\right)^{2}-\left(\frac{u}{5}\right)^{2}\right)^{\frac{1}{2}}\right] j=4
\end{aligned}
$$

And then, to easier view the rotation, the interval for $v$ is decreased to $0 \leq v \leq 5.6$, so the sphere surfaces are in cutaway:


[^7]
### 15.23 A Three Dimensional Solar System

Any number of bodies may be added to the system by specifying a geometry and a trajectory function for it, and summing them with the trajectories of the bodies around which it orbits; and then, for the third dimension, vertical, in parametric form, adding its motion to the vertical axis. E.g., for an additional, spherical 'Moon' (blue) around the planet, with an elliptical orbit at an angle to the 'planet's' orbit:

$$
\begin{gathered}
G_{M 2}=e^{4 i n}\left(e^{i v}\left(\left(\frac{1}{5}\right)^{2}-\left(\frac{u}{5}\right)^{2}\right)^{\frac{1}{2}}\right) \\
T_{M 2}=\left(k e^{l i n}+(k+1) e^{-l i n}\right) k=.3, l=3 \\
{\left[\begin{array}{c}
x=\operatorname{real}\left(G_{M 2}+T_{M 2}+T_{S}+T_{P}\right) \\
y=\frac{u}{5}+\operatorname{real} G_{M 2} \\
i z=\operatorname{imag}\left(G_{M 2}+T_{M 2}+T_{S}+T_{P}\right)
\end{array}\right]}
\end{gathered}
$$

3D orbit in blue. Others in the plane:


## Animation 79 '3D Solar System’

### 15.24 A Helical Solar System

The solar system as a whole need not only orbit a fixed point, but can follow any trajectory as in section 15.1 Objects in Polynomial Space Trajectories. If the Sun moves through space in a more or less discernible straight line, as our own Sun does, then the planets and other orbiting bodies will form helical paths.

Changing the Sun's trajectory to $t$ will allow a simple, horizontal linear movement. And then, for easier viewing, adding the animation variable $t$ to the vertical axis, which previously had no motion for the two dimensional system:

$$
T_{S}=t
$$

The vertical, $y=t+\left\{u, \frac{u}{2}, \frac{u}{5}, \frac{u}{10}\right\}$


Animation 80 'Helical Solar System’

### 15.3 Circular Waves in Motion

### 15.31 Spin Coefficient

For circular waves, the spin coefficient (section 15.22 Bodily Rotation) will generate two different types of rotation depending on the wave.

For the circular helicoid, section 14.2 Circular Helicoid:

$$
\begin{gathered}
G=e^{f(u+i v)} \\
O=b e^{i v} \\
R=e^{i v} \\
f=6, \quad b=4, \quad-4 \leq u \leq 0, \quad 0 \leq v \leq 2 \pi \\
\begin{array}{c}
x=\operatorname{realO}+\operatorname{real}(R \operatorname{imag} G) \\
y=\operatorname{real} G \\
i z=\operatorname{imag} O+\operatorname{imag}(R \operatorname{imag} G)
\end{array}
\end{gathered}
$$

adding the spin coefficient $e^{i t}$ to the geometry function:

$$
\begin{gathered}
G=e^{i t}\left(e^{f(u+i v)}\right) \\
0 \leq t \leq 20
\end{gathered}
$$

has the effect of rotating the circular helicoid around the vertical axis:
(The software resolution isn't quite high enough, so the amplitudes are a little choppy.)


Animation 81 'Rotating Circular Helicoid'

And, for the circular 'transverse wave,' section 14.4 Circular 'Transverse Wave' Surfaces:

$$
\begin{gathered}
x=\text { real } O+\text { real }(R \operatorname{imag} G) \\
y=\text { real } G \\
i z=\operatorname{imag} O+\operatorname{imag}(R \operatorname{imag} G)
\end{gathered}
$$

$$
\begin{gathered}
G=\left(1-u^{2}\right)^{\frac{1}{2}} \int_{0}^{v} e^{f i v} d v \\
O=R=e^{i v}
\end{gathered}
$$

$$
f=6,
$$

$$
0 \leq u \leq 1.1, \quad 0 \leq v \leq 2 \pi
$$

adding the spin coefficient to the geometry function:

$$
G=e^{i t}\left[\left(1-u^{2}\right)^{\frac{1}{2}} \int_{-v}^{v} e^{i v} d v\right]
$$

has the effect of rotating the amplitudes perpendicular to the circular orbit:


[^8]
### 15.32 Reciprocal Spin

Making the spin coefficient a reciprocal for the circular helicoid:

$$
G=e^{-i t}\left(e^{f(u+i v)}\right)
$$

results in rotating the circular helicoid in the opposite direction around the vertical axis:


Animation 83 'Reciprocal Rotating Circular Helicoid'

And, making the spin coefficient a reciprocal for the transverse wave:

$$
G=e^{-i t}\left[\left(1-u^{2}\right)^{\frac{1}{2}} \int_{-v}^{v} e^{i v} d v\right]
$$

results in the amplitudes rotating perpendicular to the circular orbit in the opposite direction:


Animation 84 'Reciprocal Rotating Transverse Wave’

### 15.33 Standing Waves

Adding the rotation and the reciprocal rotations together generates two different forms of standing waves:

$$
G=e^{i t}\left[\left(1-u^{2}\right)^{\frac{1}{2}} \int_{-v}^{v} e^{i v} d v\right]+e^{-i t}\left[\left(1-u^{2}\right)^{\frac{1}{2}} \int_{-v}^{v} e^{i v} d v\right] \text { Transverse }
$$



[^9]And:

$$
G=e^{i t}\left(e^{f(u+i v)}\right)+e^{-i t}\left(e^{f(u+i v)}\right) \text { Helicoid }
$$



Animation 86 'Circular Helicoid Standing Wave'

## Concluding Personal Comments

The mathematical concept of Third Dimension Imaginary is probably a good example of what Victor Hugo referred to as "an idea whose time has come." Very likely there are other individuals researching the same concept who will add much to the basic ideas.

If one accepts the concept of observable and embedded variables there is much that can be expanded upon. For example, in section 14.0 a surface is being input and, parametrically, a surface is being output with:

$$
\text { 5Dii } \quad \begin{aligned}
& x=f(u+i v) \\
& y=g(u+i v) \\
& i z=h(u+i v)
\end{aligned}
$$

Using quaternions, theoretically, a 'volume input' or a 'density input' could be made parametrically with:

$$
\text { 7Diijk } \quad \begin{aligned}
& x=f(t+i u+j v+k w) \\
& y=g(t+i u+j v+k w) \\
& i z=h(t+i u+j v+k w)
\end{aligned}
$$

If one considers Octonions, there would be eight dimensions/variables as input and three dimensions/variables as output for a total of eleven.

The point being that if the results so far, and their potential expansions are successfully inspired and realized, an "idea whose time has come," would be an apt description.

Finally, to offer one of those classically ironic anecdotes: I actually failed calculus the first two times that I took it in college! And, I went on to take it three more times, for a total of five over the next twenty years, before I felt like I understood it.

The initial inspiration for Third Dimension Imaginary came somewhere in the middle of that time period. And, when I found the orthogonal circle in between the vertices of the standard hyperbola, I knew it was correct or, at the very least, that a valid alternative coordinate system could be based on it.

## The point of that is just as I said in the dedication; It took me nearly thirty years to write this. Don't give up on what you love!

Sincerely,
greg ehmka

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[^0]:    top view 'TIP'

[^1]:    Animation 4 'Conic Nonlinearity'

[^2]:    Animation 19 'Lame Curve Morphing'

[^3]:    Animation 28 'Complex Slope and Polynomial Space Trajectory'

[^4]:    Animation 29 'Negative Base Spiral Morphing'

[^5]:    Animation 40 'Rotating Castle Tower Helix'

[^6]:    Animation 45 'Helix Antiderivative'

[^7]:    Animation 78 'Orbits and Bodily Rotations'

[^8]:    Animation 82 'Rotating Transverse Wave'

[^9]:    Animation 85 'Circular Transverse Standing Wave’

